PERFECT NUMBERS WITH IDENTICAL DIGITS

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Dedicated to Carl Pomerance on his 65th birthday.

Abstract

Suppose $g \geq 2$. A natural number N is called a *repdigit in base* g if it has the shape $a\frac{g^n-1}{g-1}$ for some $1 \leq a < g$, i.e., if all of its digits in its base g expansion are equal. The number N is called *perfect* if $\sigma(N) = 2N$, where $\sigma(N) := \sum_{d|N} d$ is the usual sum of divisors function. We show that in each base g, there are at most finitely many repdigit perfect numbers, and the set of all such numbers is effectively computable. In particular, 6 is the only repdigit perfect number in base 10.

1. Introduction

This article is about two topics which have long been of interest in recreational mathematics, repdigits and perfect numbers. A *repdigit* in base g (with $g \geq 2$) is a natural number N of the form

$$a + ag + ag^2 + \cdots + ag^{n-1} = a\frac{g^n - 1}{g - 1}, \quad \text{where } n \geq 1 \text{ and } 1 \leq a < g.$$

A *perfect number* N is a solution of the equation $\sigma(N) = 2N$, where

$$\sigma(N) = \sum_{d|N} d$$

is the usual sum of divisors function from elementary number theory.

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The arithmetic properties of repdigits are not well-understood. For example, are there infinitely many primes which are repdigits? In base 2, this is precisely the question of whether or not there are infinitely many Mersenne primes, which is an old unsolved problem. The following more approachable question about repdigits was asked by Obláth [7] in 1956: Are there any repdigits in base 10 which are perfect powers[2] other than $1, 4, 8$, and 9? He showed that there are no other examples except possibly when $a = 1$. In 1976, Shorey & Tijdeman [11] showed that in every base $g \geq 2$, there are only finitely many repdigit perfect powers, and all examples are bounded by an effectively computable constant (depending on g). In 1999, Bugeaud & Mignotte [1] settled Obláth's original problem, showing that there are no more examples in base 10 even in the remaining case $a = 1$.

In this note we consider repdigits which are also perfect numbers and prove the following two theorems:

Theorem 1. *Fix $g \geq 2$. There are only finitely many repdigit perfect numbers in base g, and the set of all such numbers is effectively computable.*

Theorem 2. *When $g = 10$, the only repdigit perfect number is* 6*.*

The proofs use well-known results on exponential Diophantine equations (extracted from [12]) and some of the ideas implicit in Luca's demonstration that there are no perfect Fibonacci or Lucas numbers ([5]; see also [6]).

Notation

Lowercase letters from the Latin alphabet, as well as N and M, always denote integers, with l, p and r reserved for primes. If $n > 1$, we write $p^-(n)$ for the smallest prime factor of n and $p^+(n)$ for its largest prime factor. When p is a prime, we write $v_p(n)$ for the exponential valuation associated to p; thus for a rational number a/b, we have $v_p(a/b) = n$ exactly when one can write $a/b = p^n c/d$, where $p \nmid cd$. We also write $\square$ to indicate a generic element of $(\mathbf{Q}^\times)^2$; thus if x and y are nonzero rational numbers, $x = y\square$ indicates that $x/y \in (\mathbf{Q}^\times)^2$, or equivalently that $x = y$ in $\mathbf{Q}^\times/(\mathbf{Q}^\times)^2$. Other notation is either standard or will be introduced as necessary.

2. Preliminary results

Fix $g \geq 2$. We write U_n and V_n for the Lucas sequences with parameters $P = g + 1$ and $Q = g$, so that

$$U_n = \frac{g^n - 1}{g - 1} \quad \text{and} \quad V_n = g^n + 1.$$

[2]Of course the word *perfect* in *perfect powers* has nothing to do with *perfect numbers*. We attempt to avoid all possible ambiguity in the sequel by referring to 'squares' instead of 'perfect squares', etc.

For basic facts about Lucas sequences, see, e.g., [9, Chapter 1] or [13, Chapter 4]. The following result on congruences is contained in Lemmas 6.2.1 and 6.4A.1 of [10]:

Lemma 1. *Suppose $g \geq 2$, and let l be a prime dividing $g-1$. If $l > 2$, then for each positive integer n,*

$$v_l(U_n) = v_l(n).$$

If $l = 2$, equality holds if either $l \nmid n$ or if $l^2 \mid g-1$.

The next lemma combines two famous 18th century results of Euler:

Lemma 2 (Euler)**.** *Suppose that N is a perfect number.*

(i) If N is even, then one can write $N = 2^{p-1}(2^p - 1)$, where both p and $2^p - 1$ are prime.

(ii) If N is odd, then one can write $N = rs^2$ where s is a natural number and r is a prime with $r \equiv 1 \pmod 4$.

Note that the converse of Lemma 2(i) is true (as was known already to Euclid) while the converse of (ii) is false.

Lemma 3 (Ljunggren [3])**.** *The only integer solutions (x, n, y) with $|x| > 1$, $n > 2$, and $y > 0$ to the exponential Diophantine equation*

$$\frac{x^n - 1}{x - 1} = y^2$$

are $(x, n, y) = (7, 4, 20)$ and $(x, n, y) = (3, 5, 11)$.

The following lemma is a corrected form of Ribenboim's Theorem A in [8]:

Lemma 4. *Suppose $g > 2$. If n and m are positive integers with $U_n U_m = \square$, then either both $U_n = \square$ and $U_m = \square$, or $n = m$. If $g = 2$, then this result is still true with a single exception (up to reordering), namely $U_3 U_6 = 7 \cdot 63 = 21^2$.*

Proof. Write $d = \gcd(n, m)$. Then $\gcd(U_n, U_m) = U_d$ (see [9, p. 9] or [13, Corollary 4.3.5]). Since $\frac{U_n}{U_d}\frac{U_m}{U_d} = \square$ and $\gcd(U_n/U_d, U_m/U_d) = 1$, it follows that

$$\frac{U_n}{U_d} = \frac{(g^d)^{n/d} - 1}{g^d - 1} = \square \quad \text{and} \quad \frac{U_m}{U_d} = \frac{(g^d)^{m/d} - 1}{g^d - 1} = \square. \tag{1}$$

By Lemma 3, either both n/d and m/d belong to the set $\{1, 2\}$, or $g^d \in \{3, 7\}$. In the latter case, $d = 1 = U_d$, and so both U_n and U_m are squares by (1).

In the former case, since n/d and m/d are relatively prime, either $n/d = 1$ or $m/d = 1$, or both. If $n/d = m/d = 1$, then $n = m$ and we are done. Finally (interchanging n and m if necessary) we may suppose $n/d = 1$ while $m/d = 2$. Then $U_m/U_d = g^d + 1$ is a square. We may suppose $d > 1$ for the same reason as in the last paragraph, so that g^d is a proper power. By a result of Ko [2] towards Catalan's conjecture, it follows that $g^d = 8$, so that $g = 2$ and $d = 3$. Then $n = d = 3$ and $m = 2d = 6$. $\square$

The next two results depend on Baker's theory of linear forms in logarithms. We begin with a special case of [12, Theorem 9.6]. If S is a finite (possibly empty) set of rational primes, we call a natural number A an *S-number* if A is supported on the primes in S.

Lemma 5. *Let $g \geq 2$. Let S be a finite (possibly empty) set of prime numbers. The set of n for which $U_n = A\Box$ for some S-number A is a finite set. Moreover, all such n are bounded by an effective constant depending only on g and S. The same holds with V_n in place of U_n.*

The next result is a special case of [12, Theorem 6.2].

Lemma 6. *Let $f(T) \in \mathbf{Q}[T]$ be a polynomial with at least three simple zeros. If b is a given nonzero rational number, then there are only finitely many pairs of integers x and y which satisfy*

$$f(x) = by^2.$$

Moreover, $|x|$ and $|y|$ are bounded by a computable number depending at most on b and f.

3. Proofs of Theorems 1 and 2

Proof of Theorem 1

Throughout this section we assume that $g \geq 2$ is fixed.

We begin by treating the case of even perfect numbers.

Lemma 7. *There are only finitely many repdigit numbers in base g which are even and perfect. In fact, all such numbers are strictly less than U_3, and so have at most two digits in base g.*

Proof. Suppose for the sake of contradiction that aU_n is even and perfect, where $1 \leq a < g$ and $n \geq 3$. Since every repdigit in base 2 is odd, we have $g > 2$. By Lemma 2, we may write $aU_n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both prime.

Suppose first that U_n is odd. Since $1 < U_n \mid 2^{p-1}(2^p - 1)$, it follows that $U_n = 2^p - 1$. Thus $a = 2^{p-1}$. But since $n \geq 3$,

$$g^2 < U_3 \leq U_n = 2^p - 1 < 2^p, \quad \text{whence} \quad g < 2^{p/2} \leq 2^{p-1} = a,$$

contradicting that $a < g$. If U_n is even, then since $U_n = 1 + g + \cdots + g^{n-1}$, it follows immediately that g is odd and n is even. Write $n = 2m$, and observe that

$$V_m U_m = (g^m + 1)\frac{g^m - 1}{g - 1} = U_n \mid 2^{p-1}(2^p - 1). \tag{2}$$

If 2 divides m, then $V_m = g^m + 1$ has a prime divisor from the residue class 1 (mod 4), contradicting (2). Hence $2 \nmid m$. Consequently, U_m is odd; since $m > 1$ and $2^p - 1$ is prime, (2) implies that $U_m = 2^p - 1$. Hence $V_m \mid 2^{p-1}$. So V_m is a power of 2. But

$$V_m = (g+1)(g^{m-1} - g^{m-2} + \cdots + 1);$$

the second factor here is odd, so must equal 1. Thus $V_m = g + 1 = V_1$, and so $m = 1$, which is a contradiction. □

Lemma 8. *Let M be a natural number. Suppose that N is a repdigit in base g, say $N = aU_n$, where $1 \le a < g$ and $p^-(n) \le M$. If $N = r\square$ for some prime r, then N is bounded by a computable constant depending only on g and M.*

Proof. It is enough to show that n is effectively bounded. We can assume that $r \nmid a$; otherwise $U_n = ra\square$, where ra is supported on the primes dividing $g!$, and the result follows from Lemma 5.

Now let p be the smallest prime factor of n, so that $p \le M$. Then for some natural number s,

$$aU_n = a\frac{g^{n/p} - 1}{g - 1}\left(1 + g^{n/p} + g^{2n/p} + \cdots + g^{(p-1)n/p}\right) = rs^2.$$

Let

$$d = \gcd\left(\frac{g^{n/p} - 1}{g - 1}, 1 + g^{n/p} + \cdots + g^{(p-1)n/p}\right).$$

Since $g^{n/p} \equiv 1 \pmod{d}$, it follows that

$$1 + g^{n/p} + \cdots + g^{(p-1)n/p} \equiv \overbrace{1 + \cdots + 1}^{p \text{ times}} \equiv p \pmod{d}.$$

Thus $d \mid p$. So there exist squarefree natural numbers u and v supported on the primes dividing apr with

$$\frac{g^{n/p} - 1}{g - 1} = u\square \quad \text{and} \quad 1 + g^{n/p} + g^{2n/p} + \cdots + g^{(p-1)n/p} = v\square, \tag{3}$$

with $auv = r\square$. In particular, $v_r(auv)$ is odd. Since $r \nmid a$, this implies that r divides uv. In fact, r must divide precisely one of u and v, since otherwise $v_r(auv) = 2$.

Suppose first that $r \nmid u$, so that $u \mid ap$. Since $a < g$ and $p \le M$, it follows that u is supported on the set S of primes dividing $g!M!$. Lemma 5 (with this set S) applied to the first relation in (3) implies that n/p is bounded by an effective constant depending only on g and M. Since $p \le M$, it follows that n is also bounded.

So suppose instead that $r \nmid v$, so that v is supported on the set S of primes defined above. Put $w = g^{n/p}$. Then from the second in relation in (3), we deduce that for some natural number z,

$$1 + w + w^2 + \cdots + w^{p-1} = vz^2, \quad \text{where we write} \quad w = g^{n/p}. \tag{4}$$

First suppose $p > 3$; then the left-hand side of (4) is a polynomial in w of degree $p-1$, all of whose roots are distinct. (Indeed, its roots are exactly the $\varphi(p) = p-1$ primitive pth roots of unity.) So by Lemma 6, $w = g^{n/p}$ is bounded by a computable constant depending only on p and v. Since $p \le M$ and $v \mid \prod_{p \in S} p$, it follows that n is also bounded. If $p = 3$, we observe that we can write either $w = w'^2$ or $w = gw'^2$ for some integer w'; that n is bounded now follows by applying Lemma 6 to the resulting degree 4 polynomials in w'. Finally, if $p = 2$, then the left-hand side of (4) is $V_{n/p}$, and we can apply Lemma 5 to see that n/p, and hence n, is bounded. □

Lemma 9. *Suppose that $\frac{g^n-1}{g-1} = r\Box$ for some prime r. If n is composite, then n is bounded by a computable number depending only on g.*

Proof. Put $p = p^+(n)$. By Lemma 8, we can assume $p \ge M$, where we now fix

$$M := \max\{g, 7\}. \tag{5}$$

We show below that under this assumption on p, there are no composite n as in the statement of the lemma.

For some natural number s, we have

$$\frac{g^n-1}{g^p-1}\frac{g^p-1}{g-1} = rs^2. \tag{6}$$

We claim that the two left-hand factors are relatively prime. Indeed, suppose for the sake of contradiction that l is a common prime factor, and let e be the order of g modulo l. Since $l \mid g^p - 1$, either $e = 1$ or $e = p$. If $e = 1$, then $l \mid g-1$; since $v_l\left(\frac{g^p-1}{g-1}\right) \ge 1$, we have $l \mid p$ from Lemma 1. So $p = l \mid g-1$, contradicting that $p \ge M \ge g$. So we must have $e = p$. But then $p \mid \#(\mathbf{Z}/l\mathbf{Z})^\times = l-1$, and so $l > p$. Since $l \mid \frac{g^n-1}{g^p-1}$, Lemma 1 implies that $l \mid \frac{n}{p} \mid n$, contradicting that $p = p^+(n)$. This proves the coprimality claim.

We now conclude from (6) that either $\frac{g^n-1}{g^p-1} = \Box$ or $\frac{g^p-1}{g-1} = \Box$. Since $p \ge M \ge 7$, it follows from Lemma 3 that $U_p = \frac{g^p-1}{g-1}$ is not a square. Hence $\frac{g^n-1}{g^p-1} = U_n/U_p$ is a square, and thus so is U_nU_p. Since U_p is not a square and $p \ge 7$, it follows from Lemma 4 that $n = p$. So n is prime. □

We can now handle the case of odd perfect numbers of the form aU_n, where the digit $1 \le a < g$ is a square. In particular, this includes the *repunit* case, when $a = 1$.

Lemma 10. *Suppose a is a square satisfying $1 \le a < g$ and that aU_n is perfect. Then n is bounded by an effectively computable number depending only on g.*

Proof. By Lemma 7, we can assume that aU_n is odd. So by Lemma 2, $aU_n = r\Box$ for some prime r, and hence $U_n = r\Box$. So by Lemma 9, we may assume that n is prime, say $n = p$.

We now show that one can compute an upper bound on those primes p for which aU_p is perfect.

Since a is a square, it follows from Lemma 2 that a is not perfect. Consequently,

$$|\sigma(a)/a - 2| \geq 1/a > 1/g.$$

Moreover, if aU_p is perfect for some p, then we must have $\sigma(a) < 2a$; otherwise $\sigma(aU_p) \geq 1 + \sigma(a)U_p > 2aU_p$. So we can assume that $\sigma(a)/a < 2 - 1/g$. We now show that for

$$p \geq 48^2 g^2, \tag{7}$$

we have $\sigma(U_p)/U_p \leq 1 + 1/(2g)$, so that

$$\frac{\sigma(aU_p)}{aU_p} \leq \frac{\sigma(a)}{a}\frac{\sigma(U_p)}{U_p} \leq \left(2 - \frac{1}{g}\right)\left(1 + \frac{1}{2g}\right) = 2 - \frac{1}{2g^2} < 2,$$

and hence aU_p is not perfect.

To prove the claim, suppose that $p \geq g$ and that l is a prime divisor of U_p. Let e denote the order of g modulo l. The proof of Lemma 9 shows that $e = p$ and hence $l \equiv 1 \pmod p$. So, using $\omega(U_p)$ to denote the number of distinct prime factors of U_p, we have

$$\begin{aligned}\frac{\sigma(U_p)}{U_p} &= \sum_{d|U_p}\frac{1}{d} \leq \prod_{l|U_p}\left(1 + \frac{1}{l} + \frac{1}{l^2} + \dots\right) \\ &\leq \exp\left(\sum_{l|U_p}\frac{1}{l-1}\right) \leq \exp\left(\sum_{1\leq j\leq \omega(U_p)}\frac{1}{pj}\right).\end{aligned}$$

Since $\sum_{j\leq\omega(U_p)}\frac{1}{j} \leq 1 + \log\omega(U_p)$, it follows that

$$\frac{\sigma(U_p)}{U_p} \leq \exp(1/p)\exp(\log(\omega(U_p))/p). \tag{8}$$

But trivially, $2^{\omega(U_p)} \leq U_p < g^p$, so that $\omega(U_p) < 2p\log g \leq 2p\log p$. It now follows, using some calculus and (8), that

$$\begin{aligned}\frac{\sigma(U_p)}{U_p} &\leq \left(1 + \frac{2}{p}\right)\left(1 + \frac{2\log(2p\log p)}{p}\right) \\ &\leq \left(1 + \frac{2}{p}\right)\left(1 + \frac{6\log p}{p}\right) \leq \left(1 + \frac{6\log p}{p}\right)^2,\end{aligned} \tag{9}$$

which is certainly smaller than $1 + 1/(2g)$ once $6\log p/p < 1/(8g)$. Since $\log p < \sqrt{p}$, this is implied by (7). □

We can now complete the proof of Theorem 1.

Proof of Theorem 1. Suppose that $1 \le a < g$ and aU_n is perfect. By Lemma 7, we can assume aU_n is odd, so that by Lemma 2 we have $aU_n = r\square$ for some prime r. By Lemma 10, we can assume that a is not a square. Thus, we may choose a prime $l \mid a$ for which $v_l(a)$ is odd. If $r = l$, then $U_n = al\square$, and the result follows from Lemma 5 (with S the set of primes dividing $g!$). Otherwise, from $aU_n = r\square$, we deduce that $l \mid U_n = \frac{g^n-1}{g-1}$. Let e denote the order of g modulo l, so that $1 \le e < l$. Then $e \mid n$. Moreover, if $e = 1$, then from $l \mid U_n$ we may deduce (via Lemma 1) that $l \mid n$. So in either case, n is divisible by some integer (either e or l) in the interval $(1, g-1]$. But then $p^-(n) < g$, and so n is bounded from Lemma 8. $\square$

Proof of Theorem 2

The proof of Lemma 7 shows that an even perfect number N which is a repdigit in base 10 has at most two digits. So by Lemma 2, either $N = 6$ or $N = 28$. So we may assume for the proof of Theorem 2 that N is odd. Write $N = aU_n$, where $1 \le a < 10$ is odd. If $a \in \{1, 5, 9\}$, then either $N \in \{1, 5, 9\}$ or $N \equiv 3 \pmod 4$. Since an odd perfect N must satisfy $N \equiv 1 \pmod 4$ by Lemma 2, we are reduced to considering the cases when $a = 3$ or $a = 7$.

Suppose first that $a = 3$. Put $p = p^+(n)$. Then if $3U_n$ is perfect,

$$3U_n = 3\left(\frac{10^p-1}{9}\right)\left(\frac{10^n-1}{10^p-1}\right) = rs^2 \tag{10}$$

as in Lemma 2. The argument of Lemma 9 showing that the left-hand factors of (6) are relatively prime also shows that 3 is the only prime which can divide

$$\gcd\left(3\left(\frac{10^p-1}{9}\right), \frac{10^n-1}{10^p-1}\right).$$

It now follows from (10) that one of

$$\frac{10^p-1}{9}, \quad 3\cdot\frac{10^p-1}{9}, \quad \frac{10^n-1}{10^p-1}, \quad 3\cdot\frac{10^n-1}{10^p-1}$$

is a square. The first is never a square since it belongs to the residue class $3 \pmod 4$. The second and fourth are also never squares since they belong to the residue class $3 \pmod 5$. We conclude that $\frac{10^n-1}{10^p-1} = U_n/U_p$ is a square, so that $U_nU_p = \square$. Lemma 3 implies that U_n is never a square when $n > 1$, and so by Lemma 4, we must have $n = p$. But

$$\frac{\sigma(3U_p)}{3U_p} \le \frac{4}{3}\frac{\sigma(U_p)}{U_p}.$$

Using (9) from the proof of Lemma 10, we compute that $\frac{\sigma(U_p)}{U_p} < \frac{3}{2}$ whenever $p \ge 29$, so that $\sigma(3U_p) < 2(3U_p)$ for such p. Since is easy to check that $3U_p$ is not an odd perfect number for any $p < 29$, we have the desired result in the case $a = 3$.

Suppose now that $a = 7$. Once again, putting $p = p^+(n)$, we have that if $7U_n$ is perfect, then

$$7U_n = 7\left(\frac{10^p - 1}{9}\right)\left(\frac{10^n - 1}{10^p - 1}\right) = rs^2 \tag{11}$$

as in Lemma 2. Since $r \equiv 1 \pmod 4$, we have $r \neq 7$. Put

$$d = \gcd\left(\frac{10^p - 1}{9}, \frac{10^n - 1}{10^p - 1}\right).$$

Referring back again to the argument of Lemma 9, we find that 3 is the only prime that can possibly divide d. If $d > 1$, then $3 \mid (10^p - 1)/9$, and so $p = 3$. So either $d = 1$, or $p = 3$ and d is a power of p. We will show below that $p \neq 3$, so that $d = 1$. Assuming this for the moment, we obtain from (11) that one of

$$\frac{10^p - 1}{9}, \quad 7 \cdot \frac{10^p - 1}{9}, \quad \frac{10^n - 1}{10^p - 1}, \quad 7 \cdot \frac{10^n - 1}{10^p - 1}$$

is a square. We have already seen that the first expression is never a square, and the second and fourth cannot be squares since they belong to the progression $2 \pmod 5$. The rest of the proof runs parallel to the case $a = 3$ described above.

It remains to rule out the possibility that $p = 3$. Since $r \neq 7$, the relation (11) implies that $7 \mid U_n \mid 10^n - 1$. Hence $6 \mid n$. Thus

$$3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 37 = 7 \cdot \frac{10^6 - 1}{9} \mid 7U_n = rs^2. \tag{12}$$

Since $11 \equiv 3 \pmod 4$, it follows that $r \neq 11$, and so we deduce from (12) that $11^2 \mid U_n \mid 10^n - 1$. Since the order of 10 modulo 11^2 is $2 \cdot 11$, it follows that $2 \cdot 11 \mid n$. In particular, $p = p^+(n) \geq 11$.

Concluding remarks

Call the pair (g, n) *exceptional* if there is an n-digit repdigit perfect number in base g. Clearly $(g, 1)$ is exceptional when $g > 6$. Also, if there are infinitely many perfect numbers, then "11" represents a perfect number in infinitely many bases g, and so there are infinitely many exceptional pairs of the form $(g, 2)$. We conjecture that there are no exceptional pairs (g, n) with $n \geq 3$. In view of Lemma 7, this would follow from the nonexistence of odd perfect numbers.

The *abc*-conjecture gives a partial result in this direction. Indeed, suppose that for a given positive constant $\theta < 1$, every odd perfect number N satisfies

$$\prod_{p \mid N} p \ll_\theta N^\theta. \tag{13}$$

With $\theta = 3/4$, (13) follows easily from the second half of Lemma 2, and Luca and Pomerance have recently proved (13) with $\theta = 17/26$. Assume now that $n \geq 3$, and that

$$N := a\frac{g^n - 1}{g - 1}, \quad \text{where} \quad 1 \leq a < g.$$

The *abc*-conjecture, applied to the equation $(g^n - 1) + 1 = g^n$, shows that

$$\prod_{p|N} p > \frac{1}{g} \prod_{p|g^n-1} p \gg_\delta g^{\delta n - 2} \geq N^{\delta - 2/n}, \tag{14}$$

for each δ with $2/3 < \delta < 1$. Let n_0 be the least integer satisfying

$$n_0 > 2(1 - \theta)^{-1}.$$

Then comparing (13) and (14), and taking δ sufficiently close to 1 (in terms of θ), we find that if N is perfect, then N is bounded by a constant depending only on θ. In other words, the set of exceptional pairs (g, n) with $n \geq n_0$ is finite. Taking $\theta = 3/4$ yields $n_0 = 9$, while $\theta = 17/26$ gives $n_0 = 6$.

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