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On congruences of the form $\sigma(n) \equiv a \pmod{n}$

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We study the distribution of solutions n to the congruence $\sigma(n) \equiv a \pmod{n}$. After excluding obvious families of solutions, we show that the number of these $n \le x$ is at most $x^{1/2+o(1)}$, as $x \to \infty$, uniformly for integers a with $|a| \le x^{1/4}$. As a concrete example, the number of composite solutions $n \le x$ to the congruence $\sigma(n) \equiv 1 \pmod{n}$ is at most $x^{1/2+o(1)}$. These results are analogues of theorems established for the Euler φ-function by the third-named author.

Keywords: Lehmer's totient problem; sum-of-divisors function; multiply perfect number; near-perfect number

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1. Introduction

In this paper, we continue the investigations of the third-named author [Pom75, Pom76,Pom77] into the congruences

$$n \equiv a \pmod{\varphi(n)} \tag{1.1}$$

and

$$\sigma(n) \equiv a \pmod{n}. \tag{1.2}$$

These congruences have connections to various unsolved problems in number theory. Most prominent among these is Lehmer's question [Leh32] of whether there are any composite solutions to (1.1) when $a = 1$. We also note here the many classical problems concerning multiply perfect numbers, which are those n satisfying (1.2) with $a = 0$.

It is observed in [Pom75] that for both (1.1) and (1.2), there is a natural classification of solutions as either *regular* or *sporadic*. For (1.1), there are no regular solutions unless $a > 0$ and $\varphi(a) \mid a$; in that case, we define a regular solution as a number n of the form pa, where p is a prime not dividing a. By a regular solution to (1.2), we mean a natural number n of the form

$$n = pm, \quad \text{where} \quad p \nmid m, \quad m \mid \sigma(m), \quad \text{and} \quad \sigma(m) = a. \tag{1.3}$$

(It is straightforward to check that these "regular solutions" really are solutions.) In both cases, all other solutions are called sporadic. From the prime number theorem, it is easily seen that if there are any regular solutions to (1.1) or (1.2), then the number of such up to x is $\asymp \frac{x}{\log x}$ for large x, where the implied constant depends on a. In [Pom75, Theorem 3], it is shown that sporadic solutions are much rarer: For any fixed a, the number of sporadic solutions to either (1.1) or (1.2) is at most

$$x/\exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log x \log\log x}\right), \qquad \text{as} \quad x \to \infty. \tag{1.4}$$

For solutions to the φ-congruence (1.1), the upper bound (1.4) has seen substantial improvement, first to $x^{2/3+o(1)}$ [Pom76] and soon after to $x^{1/2+o(1)}$ [Pom77] (both results are again for fixed a). There has been a fair amount of work chipping away at the size of the $o(1)$-term in the exponent of this last result [Sha85,BL07,BGN08,LP], but a new idea will be required to replace $\frac{1}{2}$ with anything smaller.

All of the results of the last paragraph apply only to (1.1) and not the σ-version (1.2). The behavior of the Euler φ-function on prime powers is much simpler than that of σ, and complications arise if one tries to mimic the arguments of [Pom76,Pom77]. Recently, the second-named author and V. Shevelev described how to overcome these difficulties for the arguments of [Pom76], proving that the sporadic solutions in $[1, x]$ to (1.2) number at most $x^{2/3+o(1)}$ [PS, Lemma 8]. (In fact, their result is *uniform* for $|a| < x^{2/3}$.) The purpose of this note is to reduce the exponent $\frac{2}{3}$ to $\frac{1}{2}$.

Theorem 1. *As $x \to \infty$, the number of sporadic solutions $n \le x$ to the congruence (1.2) is at most*

$$A(a)x^{1/2}\exp\left((2 + o(1))\sqrt{\frac{\log x}{\log\log x}}\right), \tag{1.5}$$

uniformly in integers a with $|a| \le x^{1/4}$. Here A is defined by $A(0) := 1$ and, for $a \ne 0$,

$$A(a) := \prod_{p^b \| a}\left(1 + \frac{b^2 + b}{2}\right).$$

Remark 1. The factor A in (1.5) is fairly tame, displaying similar behavior to the number of divisors of a. In particular, a theorem of Drozdova and Freĭman [DF58] (see also [Pos88, Chapter 4]) yields

$$A(a) \leq \exp\left((\log 2 + o(1))\frac{\log|a|}{\log\log|a|}\right), \quad \text{as} \quad |a| \to \infty.$$

(This should be compared with [HaWr08, Theorem 317, p. 345], where the same inequality is proved for the usual divisor function.) Hence, the bound (1.5) is indeed of the shape $x^{1/2+o(1)}$, uniformly for $|a| \leq x^{1/4}$. Thus (apart from secondary terms), (1.2) is no longer a second-class citizen compared to (1.1); both congruences are now on the same theoretical footing.

It seems plausible that the upper bound (1.5) can be replaced with a bounded power of $\log x$, even in the wider range $|a| \leq x/2$. See Remark 3(i) at the end of this paper. If this is correct, we have a long way to go!

We mention briefly an application of Theorem 1 to a problem considered in [PS]. Call a natural number n a *near-perfect number* if n is the sum of all of its proper divisors with one exception. In other words, n is near-perfect if $\sigma(n) = 2n + d$ for some proper divisor d of n. Using Theorem 1, we can show that the number of near-perfect $n \leq x$ is at most $x^{3/4+o(1)}$, as $x \to \infty$. This result with exponent $\frac{5}{6}$ appeared as Theorem 5 of [PS]. We omit the proof of our improvement, which essentially amounts to replacing [PS, Lemma 8] with the upper bound (1.5).

Notation

As above, we employ the Landau–Bachmann o and O notation, as well as the associated symbols $\ll$, $\gg$, and $\asymp$, with their usual meanings. All of our implied constants are absolute unless otherwise mentioned. The letter p, with or without subscripts, always denotes a prime variable. We write $\pi(x) = \sum_{p\leq x} 1$ for the number of primes not exceeding x. We say that d is a *unitary divisor* of n if d divides n and $\gcd(d, n/d) = 1$. If p is a prime, the notation $p^e \parallel n$ means that $p^e \mid n$ but that $p^{e+1} \nmid n$. The number of divisors of n is denoted $\tau(n)$. We use $\omega(n)$ for the number of distinct prime divisors of n and $\Omega(n)$ for the number of primes dividing n counted with multiplicity; thus, $\omega(n) = \sum_{p\mid n} 1$ and $\Omega(n) = \sum_{p^k\mid n} 1$. We write $\mathrm{rad}(n)$ for the *radical* of n, that is, the product of the distinct primes dividing n. We use $P(n)$ to denote the largest prime factor of n, with the convention that $P(1) = 1$.

2. Preparation

For the reader's convenience, we record here some simple inequalities for the *abundancy ratio* $\frac{\sigma(n)}{n}$.

Lemma 2. *All of the following hold:*

(i) For all integers $n \geq 1$, we have $\frac{\sigma(n)}{n} \leq 1 + \log n$.

(ii) If d divides n, then $\frac{\sigma(d)}{d} \le \frac{\sigma(n)}{n}$, with equality only if $d = n$.

(iii) If p_1 is the least prime dividing n, then $\frac{\sigma(n)}{n} \le \left(\frac{p_1}{p_1-1}\right)^{\omega(n)}$.

Proof. Both (i) and (ii) follow from the representation $\frac{\sigma(n)}{n} = \sum_{d\mid n} \frac{1}{d}$; we also use that $\sum_{d\mid n} \frac{1}{d} \le \sum_{d\le n} \frac{1}{d} \le 1 + \int_1^n \frac{dt}{t} = 1 + \log n$. For (iii), we observe that

$$\frac{\sigma(n)}{n} = \prod_{p^e \| n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) \le \prod_{p\mid n} \frac{p}{p-1} \le \left(\frac{p_1}{p_1 - 1}\right)^{\omega(n)}. \qquad \square$$

We also need a technical lemma from the "anatomy of integers" concerning numbers $n \le x$ with small radical. The next result is due to de Koninck and Doyon (compare with [DKD03, Théorème 4]).

Lemma 3. *For each fixed $z \ge 1$, we have (as $x \to \infty$)*

$$\sum_{d \le x^{1/z}} \sum_{\substack{e \le x \\ d\mid e,\ \mathrm{rad}(e)=\mathrm{rad}(d)}} 1 \le x^{1/z} \exp\left((2 + o(1))\sqrt{\frac{2(1-1/z)\log x}{\log\log x}}\right).$$

Actually, de Koninck and Doyon prove the upper bound of Lemma 3 for the number of $n \le x$ with $\mathrm{rad}(n) < n^{1/z}$, which is a smaller quantity than that considered in Lemma 3. However, the first step in their proof (see [DKD03, eq. (19)]) is to bound that above by the double sum

$$\sum_{d < x^{1/z}} \sum_{m < x/d,\ \mathrm{rad}(m)\mid d} 1.$$

Setting $e = md$, one can easily check that this double sum coincides with the quantity considered in our Lemma 3 (up to the inconsequential replacement of strict inequalities with non-strict ones). The rest of the proof of Lemma 3 follows the argument of de Koninck and Doyon.

In the remainder of this section, we show that a large, sporadic solution $n \le x$ to the congruence (1.2) has a divisor close to $\sqrt{x}$. The proofs are modeled on those of [Pom77]. We begin by quoting that paper's Lemma 4.

Lemma 4. *Suppose that $\delta \ge 0$, and that $0 \le a_1 \le a_2 \le \dots \le a_t$, where*

$$a_{i+1} \le \delta + \sum_{j=1}^{i} a_j \quad \text{for} \quad 1 \le i \le t-1.$$

Then for any ρ with $0 \le \rho < \sum_{i=1}^t a_i$, there is a subset $\mathscr{I}$ of $\{1, 2, 3, \dots, t\}$ for which

$$\rho - \delta - a_1 < \sum_{i \in \mathscr{I}} a_i \le \rho.$$

The following lemma establishes the σ-analogues of the assertions of [Pom77, Lemma 2(i)-(iii)]:

Lemma 5. *Suppose $a \neq 0$. Let n be a sporadic solution to the congruence $\sigma(n) \equiv a \pmod n$ with $n > 6a^2 \log(6|a|)$. Then all of the following hold:*

(i) $k := \frac{\sigma(n)-a}{n}$ is an integer at least 2,
(ii) if d is a unitary divisor of n with $d < n$, then $\frac{\sigma(d)}{d} < k$,
(iii) there is a prime $q > P(n)$ with $\frac{\sigma(nq)}{nq} > k$.

Proof. Since $kn + a = \sigma(n) \geq n > |a|$, clearly $k \geq 1$. Suppose for the sake of contradiction that $k = 1$. If n is composite with smallest prime factor p, then

$$a = \sigma(n) - n = \sum_{d|n,\ d<n} d \geq n/p \geq \sqrt{n} > |a|,$$

which is absurd. So n is prime and $a = \sigma(n) - n = 1$. But prime values of n are regular solutions to the congruence $\sigma(n) \equiv 1 \pmod n$ (satisfying (1.3) with $m = 1$), not sporadic solutions. This proves (i).

We turn now to (ii). If $a < 0$, then $\frac{\sigma(d)}{d} \leq \frac{\sigma(n)}{n} = k + a/n < k$, and so (ii) is trivial. So we may assume that $a > 0$. Now if d is a unitary divisor of n with $d < n$, then d divides n/p^e for some prime power $p^e \parallel n$. By Lemma 2(ii), we may restrict attention to the case when $d = n/p^e$. In this case,

$$\begin{aligned} a = \sigma(n) - kn &= \sigma(p^e)\sigma(d) - kdp^e \\ &= (p^e + \sigma(p^{e-1}))\sigma(d) - kdp^e = p^e(\sigma(d) - kd) + \sigma(p^{e-1})\sigma(d), \end{aligned}$$

and so

$$\sigma(d) - kd = \frac{a - \sigma(p^{e-1})\sigma(d)}{p^e}. \tag{2.1}$$

Let us show that the right-hand side of (2.1) is negative. This is easy if $e > 1$; then

$$\sigma(p^{e-1})\sigma(d) \geq p^{e-1}d \geq (p^e d)^{1/2} = n^{1/2} > a,$$

and so we have (ii) in this case. Suppose $e = 1$. If $d > a$, then the right-hand side of (2.1) is again negative, and so we again have (ii). So we can assume that $d \leq a$. Since $n = pd > a^2$, we must have $p > a$. But then the right-hand side of (2.1) is smaller than 1, and hence 0 or negative (since the left-hand side of (2.1) is an integer). But it cannot be zero, since otherwise $\sigma(d) = a = kd$, making $n = pd$ a regular solution to (1.2) instead of a sporadic one. This completes the proof of (ii).

Finally, we prove (iii). If $a > 0$, then we can take any prime $q > P(n)$. So we suppose that $a < 0$. Let $p := P(n)$ be the largest prime factor of n. We will show below that $p \leq \frac{n}{2|a|}$. Assuming this for now, we can take q to be any prime in the interval $(p, 2p]$. (Such a choice exists by Bertrand's postulate.) Indeed, since $q \leq \frac{n}{|a|}$, we have

$$\begin{aligned} \frac{\sigma(nq)}{nq} &= \left(k + \frac{a}{n}\right)\left(1 + \frac{1}{q}\right) \\ &\geq k\left(1 - \frac{|a|}{nk}\right)\left(1 + \frac{|a|}{n}\right) = k + \frac{|a|k}{n}\left(1 - \frac{1}{k} - \frac{|a|}{nk}\right). \end{aligned}$$

Since $|a| < n$ and $k \geq 2$ (by part (i) of the lemma), the final parenthesized expression is positive, and so $\frac{\sigma(nq)}{nq} > k$, as desired.

It remains to prove the upper bound on p. Suppose instead that $p > \frac{n}{2|a|}$. Write $n = p^e d$, where $p^e \parallel n$. Since $n > 4|a|^2$, we have $p > \sqrt{n}$, and so $e = 1$. By (ii), we know that $kd - \sigma(d) \geq 1$, so that from (2.1) and Lemma 2(i),

$$p = \frac{\sigma(d) - a}{kd - \sigma(d)} \leq |a| + \sigma(d) \leq |a| + d(1 + \log d) \leq |a| + 2|a|(1 + \log(2|a|)),$$

using $d = n/p < 2|a|$. It follows that

$$\begin{aligned} n = pd &< 2a^2 + 4a^2(1 + \log(2|a|)) \\ &< 6a^2(1 + \log(2|a|)) < 6a^2 \log(6|a|). \end{aligned}$$

But this contradicts the lower bound on n assumed in the lemma statement. $\square$

Lemma 6. *Suppose $a \neq 0$. Let n be a sporadic solution to the congruence $\sigma(n) \equiv a \pmod{n}$ with $n > 6a^2 \log(6|a|)$. Write*

$$n = p_1 p_2 \cdots p_t, \quad \textit{where} \quad p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_t.$$

For $0 \leq i \leq t-1$, we have

$$p_{i+1} \leq \Delta p_1 \cdots p_i, \quad \textit{where} \quad \Delta := 8(\log n)^2. \tag{2.2}$$

Proof. With $k \geq 2$ as in the statement of Lemma 5(i), fix a prime $q > P(n)$ so that $\frac{\sigma(nq)}{nq} > k$, and put $N := nq$. Now let $0 \leq i \leq t-1$. If $p_{i+1} = p_i$, then the desired inequality (2.2) is obvious (since $\Delta \geq 1$), and so we can assume that $p_{i+1} > p_i$. Then $d := p_1 \cdots p_i$ is a unitary divisor of N and the least prime factor of N/d is p_{i+1}. Recalling Lemma 2(iii), we see that

$$k < \frac{\sigma(N)}{N} = \frac{\sigma(d)}{d} \frac{\sigma(N/d)}{N/d} \leq \frac{\sigma(d)}{d} \left(\frac{p_{i+1}}{p_{i+1} - 1}\right)^{\omega(N/d)}.$$

Rearranging this inequality, we find that

$$1 - \frac{\omega(N/d)}{p_{i+1}} \leq \left(1 - \frac{1}{p_{i+1}}\right)^{\omega(N/d)} \leq \frac{\sigma(d)}{kd},$$

which after some manipulation shows that

$$\begin{aligned} p_{i+1} &\leq \omega(N/d) \left(1 - \frac{\sigma(d)}{kd}\right)^{-1} \\ &= \omega(N/d) \frac{kd}{kd - \sigma(d)} \leq \omega(N/d) kd. \end{aligned} \tag{2.3}$$

(For the last inequality, we use that $kd - \sigma(d) \geq 1$, as established in Lemma 5(ii).) Moreover, using the crude bound $\omega(n) \leq \Omega(n) \leq \frac{\log n}{\log 2}$ and Lemma 2(i), we have

$$\begin{aligned} \omega(N/d)k \leq (1+\omega(n))k &\leq \left(1 + \frac{\log n}{\log 2}\right)\left(\frac{\sigma(n)}{n} + \frac{|a|}{n}\right) \\ &\leq \left(1 + \frac{\log n}{\log 2}\right)(2 + \log n) < 8(\log n)^2 = \Delta. \end{aligned}$$

(The last inequality here follows from a simple calculation, using that $\log n > 1$.) Since $d = p_1 \cdots p_i$, the desired inequality (2.2) now follows from (2.3). □

Lemma 7. *Suppose $a \neq 0$. Let n be a sporadic solution to the congruence $\sigma(n) \equiv a \pmod{n}$ with $n > 6a^2 \log(6|a|)$. Let $x > 1$, and suppose that $n \in (x^{1/2}, x]$. Then there is a divisor d of n with*

$$\frac{x^{1/2}}{64(\log x)^4} < d \leq x^{1/2}. \tag{2.4}$$

Proof. Write $n = p_1 \cdots p_t$, with $p_1 \leq p_2 \leq \cdots \leq p_t$. Let $a_i = \log p_i$ for $1 \leq i \leq t$. Set $\delta := \log(8(\log x)^2)$. Then from (2.2), we have

$$a_{i+1} \leq \delta + \sum_{j=1}^{i} a_j \quad \text{for} \quad 0 \leq i \leq t-1.$$

In particular, taking $i = 0$, we see that $a_1 \leq \delta$. Applying Lemma 4, we obtain the existence of a subset $\mathscr{I} \subset \{1, 2, \ldots, t\}$ with

$$(\log \sqrt{x}) - 2\delta < \sum_{i \in \mathscr{I}} \log a_i \leq \log \sqrt{x}.$$

Exponentiating now gives the desired result. □

Remark 2. More generally, for any $1 \leq y \leq x^{1/2}$, the method of proof of Lemma 7 shows that n has a divisor in the interval $(y/(64(\log x)^4), y]$. Thus, the divisors of n are *dense*, in the sense of Saias [Sai97]. We will only need the case $y = x^{1/2}$, however.

3. Proof of Theorem 1

Proof of Theorem 1. If $a = 0$, then the n satisfying (1.2) are precisely the multiply perfect numbers. According to a theorem of Hornfeck and Wirsing [HoWi57, Satz 2], the number of these $n \leq x$ is $x^{o(1)}$, as $x \to \infty$, which is a much stronger bound than what is claimed in Theorem 1. So we can assume that $a \neq 0$. We can also assume that

$$n > \max\{x^{1/2}, 6a^2 \log(6|a|)\};$$

indeed, since $|a| \le x^{1/4}$, this inequality excludes $\ll x^{1/2}\log x$ values of n, which is acceptable. So by Lemma 7, there is a divisor d of n satisfying (2.4). Given d, let e be the (unique) unitary divisor of n with $\mathrm{rad}(e) = \mathrm{rad}(d)$; note that $d \mid e$. Since $e \mid n$ and $\sigma(e) \mid \sigma(n)$, (1.2) implies the simultaneous congruences

$$\sigma(n) \equiv a \pmod{e} \quad \text{and} \quad \sigma(n) \equiv 0 \pmod{\sigma(e)}.$$

By the Chinese remainder theorem, $\sigma(n)$ belongs to a uniquely determined residue class modulo $\mathrm{lcm}[e, \sigma(e)]$. Since $\sigma(n) \le n(1 + \log n) \le 2x\log x$, the number of possibilities for $\sigma(n)$, given d and e, is at most

$$\frac{2x\log x}{\mathrm{lcm}[e,\sigma(e)]} + 1.$$

Now we sum over the possible pairs d, e. Since $d \le x^{1/2}$, $d \mid e$, and $\mathrm{rad}(e) = \mathrm{rad}(d)$, we obtain from Lemma 3 (with $z = 2$) that

$$\sum_{d,e}\left(\frac{2x\log x}{\mathrm{lcm}[e,\sigma(e)]} + 1\right) \le \sum_{d,e}\frac{2x\log x}{\mathrm{lcm}[e,\sigma(e)]} + x^{1/2}\exp\left((2+o(1))\sqrt{\frac{\log x}{\log\log x}}\right). \tag{3.1}$$

The second right-hand term is acceptable for us, and so we concentrate on estimating the remaining double sum. Writing $\mathrm{lcm}[e,\sigma(e)] = \frac{e\sigma(e)}{\gcd(e,\sigma(e))}$, we see that

$$\sum_{d,e}\frac{2x\log x}{\mathrm{lcm}[e,\sigma(e)]} \le 2x\log x\sum_{d,e}\frac{\gcd(e,\sigma(e))}{e^2}.$$

Since $\sigma(n) \equiv a \pmod{n}$ and $\gcd(e,\sigma(e))$ divides both $\sigma(n)$ and n, it must be that $\gcd(e,\sigma(e))$ divides a. Moreover, if we define an arithmetic function τ' by setting

$$\tau'(m) := \sum_{\substack{g \mid m \\ \mathrm{rad}(g)=\mathrm{rad}(m)}} 1,$$

then given e, there are only $\tau'(e)$ possibilities for d. Hence, writing $u = \gcd(e,\sigma(e))$ and $e = uf$,

$$\sum_{d,e}\frac{\gcd(e,\sigma(e))}{e^2} \le \sum_{u\mid a} u \sum_{\frac{x^{1/2}}{64u(\log x)^4} < f \le x/u}\frac{\tau'(uf)}{(uf)^2} \le \sum_{u \mid a}\frac{\tau'(u)}{u}\sum_{\frac{x^{1/2}}{64u(\log x)^4} < f \le x/u}\frac{\tau(f)}{f^2}. \tag{3.2}$$

(To see the last inequality, observe that every divisor of uf with the same radical as uf can be written as the product of a divisor of u with the same radical as u, multiplied by a divisor of f, so that $\tau'(uf) \le \tau'(u)\tau(f)$.) Let $S(t) := \sum_{m \le t}\tau(m)$. It is well-known (see, for example, [HaWr08, Theorem 18, p. 347]) that $S(t) \ll t\log t$. Hence,

$$\sum_{\frac{x^{1/2}}{64u(\log x)^4} < f \le x/u}\frac{\tau(f)}{f^2} \le \int_{\frac{x^{1/2}}{64u(\log x)^4}}^{\infty}\frac{dS(t)}{t^2} \ll \int_{\frac{x^{1/2}}{64u(\log x)^4}}^{\infty}\frac{\log t}{t^2}\,dt \ll ux^{-1/2}(\log x)^5.$$

Referring back to (3.2), we see that the sum appearing on the right-hand side of (3.1) is $\ll x^{1/2}(\log x)^6 \sum_{u|a} \tau'(u)$. Since τ' is multiplicative,

$$\sum_{u|a} \tau'(u) = \prod_{p^b \| a} \left(\sum_{j=0}^{b} \tau'(p^j) \right)$$
$$= \prod_{p^b \| a} \left(1 + \frac{b^2+b}{2} \right) = A(a).$$

Taking stock, we have shown that the number of possibilities for $\sigma(n)$ is

$$\ll A(a) x^{1/2} (\log x)^6 + x^{1/2} \exp\left((2+o(1)) \sqrt{\frac{\log x}{\log\log x}} \right),$$

which we recognize as being bounded by (1.5).

Write $\sigma(n) = kn + a$, so that n is determined by $\sigma(n)$ and k. With $k = \frac{\sigma(n)-a}{n}$, we have by Lemma 2(i) and Lemma 5(i) that

$$2 \le k \le \frac{\sigma(n)}{n} + 1 \le 2 + \log n \le 2 + \log x,$$

and so there are $O(\log x)$ possibilities for k. It follows that the number of possibilities for n is also bounded by (1.5). □

Remark 3.

(i) Let $S_a(x)$ denote the number of solutions to (1.2) with $n \le x$, and write $S_a(x) = S'_a(x) + S''_a(x)$, where $S'_a(x)$ counts regular solutions and $S''_a(x)$ counts sporadic ones. We have

$$\sum_{|a| \le x/2} S_a(x) = \sum_{n \le x} \sum_{\substack{|a| \le x/2 \\ a \equiv \sigma(n) \pmod{n}}} 1 = \sum_{n \le x} \left(\frac{x}{n} + O(1) \right) = x \log x + O(x). \tag{3.3}$$

On the other hand, recalling the definition (1.3) of a regular solution, we see that

$$\sum_{|a| \le x/2} S'_a(x) \le \sum_{|a| \le x/2} \sum_{\substack{m | \sigma(m) \\ \sigma(m) = a}} \pi(x/m)$$
$$\le \sum_{\substack{m \le x/2 \\ m | \sigma(m)}} \pi(x/m) \ll \frac{x}{\log x} \sum_{\substack{m \le \sqrt{x} \\ m | \sigma(m)}} \frac{1}{m} + x \sum_{\substack{m > \sqrt{x} \\ m | \sigma(m)}} \frac{1}{m}.$$

Using partial summation in combination with the upper bound of Hornfeck and Wirsing [HoWi57, Satz 2] alluded to above, the first of the two remaining sums is absolutely bounded, while the latter is at most $x^{-1/2+o(1)}$, as $x \to \infty$.

Hence, $\sum_{|a|\le x/2} S'_a(x) \ll x/\log x$. Combining this with (3.3), we obtain an asymptotic result on the average number of sporadic solutions:

$$\frac{1}{x}\sum_{|a|\le x/2} S''_a(x) = \log x + O(1).$$

This motivates the conjecture, already appearing in the introduction, that $S''_a(x) \le (\log x)^{O(1)}$ whenever $x \ge 3$ and $|a| \le x/2$.

(ii) Since the issue of uniformity seems to have been neglected in prior studies of the φ-congruence (1.1), we point out that the proof of Theorem 1 can be adapted to establish an upper bound of the form $\tau(|a|)x^{1/2}(\log x)^{O(1)}$ for the number of sporadic solutions $n \le x$ to (1.1), uniformly for $0 < |a| \le x^{1/4}$.

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