Remarks on a paper of Ballot and Luca concerning prime divisors of $a^{f(n)} - 1$

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Abstract. Let a be an integer with $|a| > 1$. Let $f(T) \in \mathbf{Q}[T]$ be a nonconstant, integer-valued polynomial with positive leading term, and suppose that there are infinitely many primes p for which f does not possess a root modulo p. Under these hypotheses, Ballot and Luca showed that almost all primes p do not divide any number of the form $a^{f(n)} - 1$. More precisely, assuming the Generalized Riemann Hypothesis (GRH), their argument gives that the number of primes $p \le x$ which do divide numbers of the form $a^{f(n)} - 1$ is at most (as $x \to \infty$)

$$\frac{\pi(x)}{(\log\log x)^{r_f + o(1)}},$$

where r_f is the density of primes p for which the congruence $f(n) \equiv 0 \pmod{p}$ is insoluble. Under GRH, we improve this upper bound to $\ll x(\log x)^{-1-r_f}$, which we believe is the correct order of magnitude.

Contents

1. Introduction 1
2. Sieving the numbers $\ell(p)$ 3
3. The case when $\mathscr{P}$ is infinite: Proof of Theorem 1 6
4. The case when $\mathscr{P}$ is finite: Proof of Theorem 2 7
5. An exercise in heuristic reasoning 9
6. Concluding remarks 12
Acknowledgements 13
References 13

1. Introduction

Fix an integer a with $|a| > 1$. From Fermat's little theorem, we know that the set of primes which divide $a^n - 1$ for some n is precisely the set of primes not dividing a. Luca and Ballot [1] investigated what happens if we replace the exponent n here by a different

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polynomial expression in n: Fix a nonconstant, integer-valued polynomial $f(T) \in \mathbf{Q}[T]$ with positive leading coefficient. Define

$$\mathscr{P} := \{q : q \text{ prime}, f(n) \equiv 0 \pmod{q} \text{ has no solution}\}. \tag{1}$$

By the Chebotarev density theorem (see, e.g., [16]), the set $\mathscr{P}$ has a Dirichlet density; call this r_f. The following is the main result of [1]; we write GRH for the *Generalized Riemann Hypothesis*, which for us is the assertion that the nontrivial zeros of all Dedekind zeta functions lie on the line $\Re(s) = \frac{1}{2}$.

Theorem A. *Assume that f is irreducible of degree > 1. Then the number of primes $p \leq x$ which divide some number of the form $a^{f(n)} - 1$, where $n \in \mathbf{N}$, is at most*

$$\pi(x)/(\log\log\log x)^{r_f + o(1)},$$

as $x \to \infty$. Assuming GRH, the upper bound can be improved to $\pi(x)/(\log\log x)^{r_f+o(1)}$.

A careful reading of the proof of Theorem A reveals that the stated estimates hold for all f, and that irreducibility is used only to guarantee that $r_f > 0$; see [1, Lemma 3]. (Of course, the estimates are trivial if $r_f = 0$.) By the density theorems in [16], one has $r_f > 0$ exactly when $\mathscr{P}$ is infinite. So as long as infinitely many primes do not divide values of $f(n)$, almost all primes (all but $o(\pi(x))$ of those in $[2, x]$, as $x \to \infty$) do not divide any expression of the form $a^{f(n)} - 1$. Moreover, replacing the use of inclusion-exclusion in the argument of [1] with a more powerful sieve, one quickly obtains an unconditional proof of the upper bound claimed under GRH. In fact, one gets an upper bound that is $\ll_a \pi(x)/(\log\log x)^{r_f}$; notice that we have removed the $o(1)$ in the exponent. See the remark at the end of §2.

By a different method, we shall improve the conditional upper bound substantially:

Theorem 1. *Assume GRH. Let a be an integer with $|a| > 1$. Suppose that the set $\mathscr{P}$ defined in* (1) *is infinite, with Dirichlet density $r_f > 0$. For $x \geq 2$, the number of $p \leq x$ dividing some $a^{f(n)} - 1$ is $\ll_{a,f} x/(\log x)^{1+r_f}$.*

Remark. For later use, it will be helpful to observe that by the Chebotarev density theorem, f splits into linear factors modulo p for a set of primes p of positive density. Thus, $r_f < 1$ always.

Theorem 1 leaves open the question of what happens when $\mathscr{P}$ is finite. This turns out to be much simpler; indeed, we can establish an asymptotic formula.

Theorem 2. *If $\mathscr{P}$ is finite, then the set of primes dividing some $a^{f(n)} - 1$ possesses a positive relative density. In other words, the number of such $p \leq x$ is $\sim c_{a,f}\pi(x)$, as $x \to \infty$, for some constant $c_{a,f} > 0$.*

We prove Theorem 2 in §4. There we also give a formula for $c_{a,f}$ when $a > 0$, using explicit results of Wiertelak [19] (cf. Pappalardi [13], Moree [10]) concerning how often a given integer d divides the order of a mod p.

It seems difficult to prove a corresponding asymptotic formula in the case when $\mathscr{P}$ is infinite. On the basis of our work in §4, we propose such a formula in §5 (again, assuming

$a > 0$). One consequence of this formula is that the primes p dividing some $a^{f(n)} - 1$ should have counting function asymptotic to a constant multiple of $x/(\log x)^{1+r_f}$. In §6, we conclude the paper with a discussion of the difficulties associated with proving a lower bound of the expected order of magnitude.

Notation. The unitalicized letter e denotes the base of the natural logarithm. We write ζ_m for the primitive mth root of unity $\mathrm{e}^{2\pi i/m}$. The letters p and q are reserved for primes. We use Erdős's notation $\ell_a(m)$ for the order of a modulo m; if a is understood, we often omit the subscript. We write $\omega(n) := \sum_{p \mid n} 1$ for the number of distinct prime factors of n. The notation $d \parallel n$ means that d is a *unitary* divisor of n, i.e., $d \mid n$ and $\gcd(d, n/d) = 1$. We employ the Landau–Bachmann O and o symbols, as well as Vinogradov's $\ll$ notation, with subscripts indicating any dependence of implied constants. We use Li for the usual *logarithmic integral*, so that $\mathrm{Li}(x) := \int_2^x dt/\log t$.

2. Sieving the numbers $\ell(p)$

Fix an integer a with $|a| > 1$. In this section, we prove an upper bound on the proportion of the time that $\ell(p)$ has a prime factor belonging to a prescribed set $\mathcal{Q}$. It seems that this result may be of some independent interest.

Theorem 3. *Assume GRH. Let $x \geq 2$, and let $\mathcal{Q}$ be a set of primes contained in $[2, x]$. The number of primes $p \leq x$ for which $\ell(p)$ is not divisible by any $q \in \mathcal{Q}$ is*

$$\ll_a \pi(x) \prod_{q \in \mathcal{Q}} (1 - 1/q), \tag{2}$$

uniformly in $\mathcal{Q}$ and x.

Remarks.

(i) As we will see in Theorem C below, apart from $O_a(1)$ exceptional primes q, the probability that q divides $\ell(p)$ is $q/(q^2 - 1)$. So from a psychological standpoint, it would appear more natural if the factors on the right-hand side of (2) were $1 - q/(q^2 - 1)$. However, replacing each term $1 - 1/q$ with the more cumbersome factor $1 - q/(q^2 - 1)$ would not change the magnitude of the right-hand side, and so would not affect the result. We have chosen to allow typography to trump psychology.

(ii) From Theorem 3, it is simple to deduce a (GRH-conditional) theorem of Murata and Pomerance [12, Theorem 4]: For $x \geq 2$, the number of odd primes $p \leq x$ for which $\ell_2(p)$ is prime is $\ll x/(\log x)^2$. (Briefly, take $\mathcal{Q}$ to be the set of primes $\leq x^{1/3}$, say, and recall that there are $o(x/(\log x)^2)$ primes $p \leq x$ with $\ell_2(p) \leq x^{1/3}$.) Our proof is similar in spirit to theirs.

Our argument rests on Lagarias and Odlyzko's explicit Chebotarev density theorem (on GRH) [8], as formulated by Serre [15, §2.4]:

Theorem B. *Assume GRH. Let K be a finite Galois extension of $\mathbf{Q}$ with Galois group G, and let C be a conjugacy class of G. The number of unramified primes $p \le x$ whose Frobenius conjugacy class $(p, K/\mathbf{Q}) = C$ is given by*

$$\frac{\#C}{\#G}\mathrm{Li}(x) + O\left(\frac{\#C}{\#G}x^{1/2}(\log|\Delta_K| + [K:\mathbf{Q}]\log x)\right),$$

for all $x \ge 2$. Here Δ_K denotes the discriminant of K and the O-constant is absolute.

We also need an estimate extracted from Hooley's GRH-conditional proof of Artin's primitive root conjecture [5].

Lemma 4. *Assume GRH. Let $x \ge 2$. There are $\ll_a x/(\log x)^2$ primes $p \le x$ which have the following property: For some prime $q \in (\log x, x^{1/2}(\log x)^{-2}]$,*

$$q \mid p-1 \quad \textit{and} \quad a^{\frac{p-1}{q}} \equiv 1 \pmod{p}.$$

Remark. Hooley's aim is to prove Artin's conjecture, and so he assumes from the start that a is not a perfect square. But Lemma 4 is valid without that restriction. It is enough that the number of $p \le x$ which split completely in $K := \mathbf{Q}(\zeta_q, a^{1/q})$ is $\frac{\mathrm{Li}(x)}{[K:\mathbf{Q}]} + O_a(x^{1/2}\log(qx))$ and that $[K:\mathbf{Q}] \gg_a q\phi(q)$. This much holds without assuming that a is not a square (cf. the argument for Theorem 3 below).

Finally, we need a known estimate on the distribution of smooth numbers. Recall that a natural number n is said to be *y-smooth* if every prime divisor p of n satisfies $p \le y$. We let $\Psi(x,y)$ denote the number of y-smooth natural numbers $n \le x$.

Lemma 5. *Fix a real number $A \ge 1$. Then $\Psi(x, (\log x)^A) = x^{1-\frac{1}{A}+o(1)}$, as $x \to \infty$.*

For a proof of Lemma 5, see, e.g., [3, p. 291].

Proof of Theorem 3. There is no loss in assuming that $\mathscr{Q} \subset [2, x^{1/2}(\log x)^{-2}]$, since $\prod_{x^{1/2}(\log x)^{-2} < q \le x}(1-1/q) \asymp 1$. Let $p \le x$ be a prime for which $\ell(p)$ is coprime to the members of $\mathscr{Q}$. The right-hand side of (2) is always $\gg x/(\log x)^2$, and so we can assume that p is not in the exceptional set considered in Lemma 4. Thus, if $q \in \mathscr{Q}$ is a divisor of $p-1$ with $q > \log x$, then $a^{(p-1)/q} \not\equiv 1 \pmod{p}$. Let M be the largest divisor of $p-1$ supported on primes belonging to $\mathscr{Q}$. Since $\ell(p)$ is coprime to the members of $\mathscr{Q}$, we must have $a^{(p-1)/M} \equiv 1 \pmod{p}$. It follows that M is supported entirely on primes not exceeding $\log x$.

We may assume that M does not exceed $\exp(\sqrt{\log x})$. Indeed, the total number of integers in $[1,x]$ divisible by some $(\log x)$-smooth integer $M > \exp(\sqrt{\log x})$ is at most

$$\sum_{\substack{\exp(\sqrt{\log x}) < M \le x \\ p \mid M \Rightarrow p \le \log x}} \left\lfloor \frac{x}{M} \right\rfloor \le x \int_{\exp(\sqrt{\log x})}^{x} \frac{d\Psi(t, \log x)}{t}. \tag{3}$$

When $t \ge \exp(\sqrt{\log x})$, we have $\log x \le (\log t)^2$, and so $\Psi(t, \log x) \ll t^{2/3}$, say, by taking $A = 2$ in Lemma 5. Hence, the right-hand side of (3) is $\ll x/\exp(\frac{1}{3}\sqrt{\log x})$. This is negligible in comparison with the upper bound in the theorem statement.

We now fix a $(\log x)$-smooth integer $M \le \exp(\sqrt{\log x})$ and use Selberg's Λ^2-sieve to count the number of corresponding $p \le x$. Let

$$\mathscr{A} := \{p-1 : p \le x, M \mid p-1, a^{\frac{p-1}{M}} \equiv 1 \pmod{p}\} \quad \text{and} \quad \mathscr{Q}' := \{q \in \mathscr{Q} : q \nmid aM\}.$$

Then the number of $p \le x$ corresponding to M is bounded above by

$$S(\mathscr{A}, \mathscr{Q}') := \#\{A \in \mathscr{A} : \gcd(A, \prod_{q \in \mathscr{Q}'} q) = 1\}.$$

We turn next to the preliminary estimates needed to apply the sieve.

Let $p \le x$ be a prime not dividing $2a$. From a well-known theorem of Kummer–Dedekind, $p-1 \in \mathscr{A}$ precisely when p splits completely in $K_1 := \mathbf{Q}(\zeta_M, a^{1/M})$. From [18, Proposition 4.1], we have $[K_1 : \mathbf{Q}] \asymp_a M\phi(M)$. Since the discriminant of $\mathbf{Q}(\zeta_M)$ divides $M^{\phi(M)}$ and the discriminant of $\mathbf{Q}(a^{1/M})$ divides $(aM)^M$, we obtain from the relation

$$\Delta_{K_1} \mid \Delta_{\mathbf{Q}(a^{1/M})}^{[K_1:\mathbf{Q}(a^{1/M})]} \Delta_{\mathbf{Q}(\zeta_M)}^{[K_1:\mathbf{Q}(\zeta_M)]}$$

(cf. [14, p. 218, Proof of 7Q]) that

$$\begin{aligned} \log|\Delta_{K_1}| &\le M\phi(M)\log(|a|M) + M\phi(M)\log M \\ &\ll_a M\phi(M)\log(\mathrm{e}M). \end{aligned}$$

So setting $X := \frac{\mathrm{Li}(x)}{[K_1:\mathbf{Q}]}$, Theorem B yields

$$\#\mathscr{A} := X + O_a(x^{1/2}\log(Mx)) = X + O_a(x^{1/2}\log x).$$

Next, let d be a squarefree natural number supported on primes belonging to $\mathscr{Q}'$. Set $\mathscr{A}_d := \{A \in \mathscr{A} : d \mid A\}$. If $p \le x$ is a prime not dividing $2a$, then $p-1 \in \mathscr{A}_d$ precisely when p splits completely in $K_2 := \mathbf{Q}(\zeta_{dM}, a^{1/M})$. View K_2 as the compositum of K_1 and $L := \mathbf{Q}(\zeta_d)$. The discriminant of L divides $d^{\phi(d)}$, while the discriminant of K_1 is supported on primes dividing aM. Hence, $\gcd(\Delta_L, \Delta_{K_1}) = 1$. We deduce that

$$[K_2 : \mathbf{Q}] = [L : \mathbf{Q}][K_1 : \mathbf{Q}] = \phi(d)[K_1 : \mathbf{Q}]$$

and

$$\Delta_{K_2} = \Delta_{K_1}^{[L:\mathbf{Q}]}\Delta_L^{[K_1:\mathbf{Q}]},$$

so that

$$\begin{aligned} \log|\Delta_{K_2}| &\ll_a \phi(d)\log|\Delta_{K_1}| + M\phi(M)\log|\Delta_L| \\ &\ll_a \phi(d)M\phi(M)\log(\mathrm{e}M) + (M\phi(M))(\phi(d)\log d) \\ &\ll M\phi(dM)\log(\mathrm{e}dM). \end{aligned}$$

Applying Theorem B again, we find that

$$\begin{aligned} \#\mathscr{A}_d &= \frac{\mathrm{Li}(x)}{\phi(d)[K_1 : \mathbf{Q}]} + O_a\left(x^{1/2}\log x + x^{1/2}\log(\mathrm{e}dM)\right) \\ &= \frac{X}{\phi(d)} + O_a(x^{1/2}\log x), \end{aligned}$$

assuming $d \le x$ (say).

Selberg's upper bound sieve, in the form of [4, p. 133, Theorem 4.1], now yields that for $z := x^{1/5}$,

$$S(\mathscr{A}, \mathscr{Q}') \ll_a X \prod_{q \in \mathscr{Q}' \cap [2,z]} \left(1 - \frac{1}{\phi(q)}\right) + x^{1/2} \log x \sum_{\substack{d \le z^2 \\ p \mid d \Rightarrow p \in \mathscr{Q}' \\ d \text{ squarefree}}} 3^{\omega(d)}. \tag{4}$$

Using the universal upper bound $\omega(d) \ll \log d / \log\log(3d)$ and recalling the restriction $d \le z^2$, we see that $3^{\omega(d)} \ll x^{1/25}$, say. So the second term on the right-hand side of (4) is $\ll x^{0.95}$. Also,

$$\begin{aligned} X \prod_{q \in \mathscr{Q}' \cap [2,z]} \left(1 - \frac{1}{\phi(q)}\right) &\ll_a \frac{\mathrm{Li}(x)}{M\phi(M)} \prod_{q \in \mathscr{Q}'} \left(1 - \frac{1}{q}\right) \\ &= \frac{\mathrm{Li}(x)}{\phi(M)^2} \prod_{q \mid M} \left(1 - \frac{1}{q}\right) \prod_{q \in \mathscr{Q}'} \left(1 - \frac{1}{q}\right) \\ &\ll_a \frac{\pi(x)}{\phi(M)^2} \prod_{q \in \mathscr{Q}} \left(1 - \frac{1}{q}\right). \end{aligned}$$

Hence, the number of $p \le x$ corresponding to M is

$$\ll_a \frac{\pi(x)}{\phi(M)^2} \prod_{q \in \mathscr{Q}} \left(1 - \frac{1}{q}\right) + x^{0.95}.$$

Now sum over all $(\log x)$-smooth values of $M \le \exp(\sqrt{\log x})$. Since the infinite series $\sum_{M \ge 1} \frac{1}{\phi(M)^2}$ converges, and since we are summing over only $x^{o(1)}$ values of M, we obtain the estimate of the theorem. □

Remark. The idea of [1] is to sieve directly the sequence $\mathscr{A} := \{\ell(p)\}_{p \le x}$, where the requisite information on the number of terms of $\mathscr{A}$ divisible by a given d can be read off from a theorem of Pappalardi [13, Theorem 1.3]. That approach, in conjunction with the same form of Selberg's sieve employed above, gives an unconditional proof of Theorem 3 under the severe restriction that $\mathscr{Q} \subset [2, \log x]$.

3. The case when $\mathscr{P}$ is infinite: Proof of Theorem 1

Assume that a and $f(T)$ satisfy the hypotheses of Theorem 1. If $p \mid a^{f(n)} - 1$ for some n, then $\ell(p) \mid f(n)$, and so $\ell(p)$ cannot be divisible by any of the primes from the set $\mathscr{P}$ defined in (1). Applying Theorem B to the splitting field of f, we find that (on GRH) the counting function of $\mathscr{P}$ behaves like $r_f \cdot \mathrm{Li}(x)$ up to an error of $O_f(x^{1/2} \log x)$. By partial summation,

$$\sum_{q \in \mathscr{P} \cap [2,x]} \frac{1}{q} = r_f \log\log x + O_f(1). \tag{5}$$

(One could also prove this last estimate unconditionally, using, e.g., [15, Théorème 2].) Theorem 1 now follows from Theorem 3 with $\mathscr{Q}$ taken as $\mathscr{P}\cap[2,x]$.

4. The case when $\mathscr{P}$ is finite: Proof of Theorem 2

We start by quoting a weakened form of a result of Wiertelak [19, Theorem 2] (see also Pappalardi [13, Theorem 1], whose notation is more similar to ours).

Theorem C. *Fix an integer a with $a>1$. Write $a=b^h$, with b not a perfect power, and put $b=a_1a_2^2$, where a_1 is squarefree. Let d be a fixed natural number. For $x\geq 3$, the number of primes $p\leq x$ for which d divides $\ell_a(p)$ is*

$$\left(\frac{\nu_{a,d}}{d(h,d^\infty)}\prod_{q\mid d}\frac{q^2}{q^2-1}\right)\mathrm{Li}(x)+O_{a,d}\left(\frac{\mathrm{Li}(x)}{(\log x)^{1.9}}\right).$$

Here (h,d^∞) is the largest divisor of h supported on the primes dividing d, and

$$\nu_{a,d}:=\begin{cases}1 & \text{if } [2,a_1]\nmid d,\\ 1/2 & \text{if } [2,a_1]\mid d,\ a_1\equiv 1 \pmod 4,\\ 1/2 & \text{if } [2,a_1]\mid d,\ a_1\not\equiv 1 \pmod 4,\ 4(2,a_1)\mid dh,\\ 5/4 & \text{if } [2,a_1]\mid d,\ a_1\not\equiv 1 \pmod 4,\ 2(2,a_1)\parallel dh,\\ 17/16 & \text{if } [2,a_1]\mid d,\ a_1\not\equiv 1 \pmod 4,\ 2(2,a_1)\nmid dh.\end{cases}$$

Remark. It follows from Theorem C that for fixed positive integers a and d with $a>1$, the primes p for which d divides $\ell_a(p)$ possess a relative density. This holds also if $a<-1$. To see this, first note that except in the case when $2\parallel d$, one has that $d\mid\ell_a(p)$ precisely when $d\mid\ell_{-a}(p)$. If $2\parallel d$, then it is easy to show that

$$\begin{aligned}\#\{p\leq x: p\nmid 2a, d\mid\ell_a(p)\} &= \#\{p\leq x: p\nmid 2a, \tfrac{d}{2}\mid\ell_{-a}(p)\}\\ &\quad+\#\{p\leq x: p\nmid 2a, 2d\mid\ell_{-a}(p)\}-\#\{p\leq x: p\nmid 2a, d\mid\ell_{-a}(p)\};\end{aligned}$$

see, e.g., [19, p. 181]. Theorem C applies to estimate all three right-hand terms and so gives the relative density in this case also. Alternatively, one can consult [10, Theorem 2], which gives expressions for the density valid regardless of the sign of a.

Proof of the existence of the density in Theorem 2. Let $\mathscr{Q}$ denote the set of primes q for which not all of the congruences $f(n)\equiv 0 \pmod{q^e}$, with $e=0,1,2,\dots$, are solvable. By Hensel's lemma, $\mathscr{Q}\setminus\mathscr{P}$ is finite, and so our assumption that $\mathscr{P}$ is finite gives that $\mathscr{Q}$ is also finite.

For each $q\in\mathscr{Q}$, there is a least positive integer e_q (say) for which the congruence $f(n)\equiv 0 \pmod{q^{e_q}}$ is insoluble. A prime p divides $a^{f(n)}-1$ for some n precisely when no prime power of the form q^{e_q}, with $q\in\mathscr{Q}$, divides $\ell(p)$. That the set of such primes p possesses a relative density now follows immediately from inclusion-exclusion and the remark following Theorem C. $\square$

It remains to show that the density whose existence was just proved is > 0. We will give an explicit expression for this density from which positivity follows by a straightforward check. Complete details are given only in the case when $a > 0$; the case when $a < 0$ presents additional difficulties which we remark on at the end.

So suppose now that $a > 1$. We may assume that a is not a perfect power, since if $a = b^h$, then $a^{f(n)} - 1 = b^{h \cdot f(n)} - 1$, and we could replace a by b and f by hf. Thus, in the notation of Theorem C, we have $h = 1$ and $a = b$.

Let $\mathscr{Q}$ denote the set introduced in the existence proof, and let $Q := \prod_{q \in \mathscr{Q}} q^{e_q}$. Inclusion-exclusion shows that our relative density is given by

$$c_{a,f} := \sum_{d \| Q} (-1)^{\omega(d)} \frac{\nu_{a,d}}{d} \prod_{q \mid d} \frac{q^2}{q^2 - 1}, \tag{6}$$

in the notation of Theorem C. If $[2, a_1] \nmid Q$, then each $\nu_{a,d} = 1$, and the sum admits the product expansion

$$\prod_{q \mid Q} \left(1 - \frac{q^2}{q^{e_q}(q^2 - 1)}\right).$$

Suppose now that $[2, a_1] \mid Q$. Write $Q = Q_1 Q_2$, where Q_1 is supported on the primes dividing $2a_1$. For unitary divisors d of Q, we see that $[2, a_1] \mid d$ if and only if $Q_1 \mid d$. This suggests splitting the sum in (6) into two pieces, $\sum_1$ and $\sum_2$, with $\sum_1$ corresponding to those d not divisible by Q_1 and $\sum_2$ corresponding to the remaining d. From $\sum_1$, we get a contribution of

$$\sum_{d \| Q} \frac{(-1)^{\omega(d)}}{d} \prod_{q \mid d} \frac{q^2}{q^2 - 1} - \sum_{\substack{d \| Q \\ Q_1 \mid d}} \frac{(-1)^{\omega(d)}}{d} \prod_{q \mid d} \frac{q^2}{q^2 - 1}$$

$$= \prod_{q \mid Q} \left(1 - \frac{q^2}{q^{e_q}(q^2 - 1)}\right) - (-1)^{\omega(Q_1)} \left(\prod_{q \mid Q_1} \frac{q^2}{q^{e_q}(q^2 - 1)}\right) \left(\prod_{q \mid Q_2} \left(1 - \frac{q^2}{q^{e_q}(q^2 - 1)}\right)\right).$$

It remains to treat $\sum_2$, corresponding to unitary divisors d of Q for which $Q_1 \mid d$. The key observation is that $\nu_{a,d}$ is constant for such d. In fact, putting

$$\nu := \begin{cases} 1/2 & \text{if } a_1 \equiv 1 \pmod 4, \\ 1/2 & \text{if } a_1 \not\equiv 1 \pmod 4,\ 4(2, a_1) \mid Q_1, \\ 5/4 & \text{if } a_1 \not\equiv 1 \pmod 4,\ 2(2, a_1) \| Q_1, \\ 17/16 & \text{if } a_1 \not\equiv 1 \pmod 4,\ 2(2, a_1) \nmid Q_1, \end{cases} \tag{7}$$

we have $\nu_{a,d} = \nu$ for all these d. Reasoning as above, we obtain a contribution from $\sum_2$ of

$$\nu \cdot (-1)^{\omega(Q_1)} \left(\prod_{q \mid Q_1} \frac{q^2}{q^{e_q}(q^2 - 1)}\right) \left(\prod_{q \mid Q_2} \left(1 - \frac{q^2}{q^{e_q}(q^2 - 1)}\right)\right).$$

Collecting the contributions from $\sum_1$ and $\sum_2$, we find that $c_{a,f}$ is equal to

$$\prod_{q|Q}\left(1-\frac{q^2}{q^{e_q}(q^2-1)}\right)$$
$$+(-1)^{\omega(Q_1)}(\nu-1)\left(\prod_{q|Q_1}\frac{q^2}{q^{e_q}(q^2-1)}\right)\left(\prod_{q|Q_2}\left(1-\frac{q^2}{q^{e_q}(q^2-1)}\right)\right).$$

Factoring out the first product appearing here, we complete the proof of the following proposition:

Proposition 6. *Assume $a>1$ and not a perfect power. Then the constant $c_{a,f}$ in Theorem 2 is given by*

$$\left(1+(\nu-1)(-1)^{\omega(Q_1)}\prod_{q|Q_1}\frac{q}{q^{e_q+1}-q-q^{e_q-1}}\right)\prod_{q|Q}\left(1-\frac{q^2}{q^{e_q}(q^2-1)}\right). \tag{8}$$

Here we take $\kappa=1$ if $[2,a_1]\nmid Q$.

Recalling the way the value of ν was selected, it is now straightforward to check directly that $c_{a,f}>0$ in the cases when $a>1$.

Suppose now that $a<-1$. If 2 is not a unitary divisor of Q, then the situation is fairly simple: For $q\in\mathscr{Q}$, the number $\ell_a(p)$ is divisible by q^{e_q} precisely when the same is true for $\ell_{-a}(p)$. So replacing a with $-a$, we may derive an expression for $c_{a,f}$ analogous to that in Proposition 6 by essentially an identical argument. (We cannot assume now that $h=1$, since $-a$ may be a perfect power, but the extra factor (h,d^∞), being multiplicative in d, does not cause any real difficulties.) Suppose now that $2\parallel Q$, so that $2\in\mathscr{Q}$ and $e_2=1$. Then we observe that

$$\#\{p\le x: p\nmid 2a, \ell_a(p) \text{ not divisible by any } q^{e_q}\}=$$
$$\#\{p\le x: p\nmid 2a, \ell_{a^2}(p) \text{ not divisible by any } q^{e_q}\}$$
$$-\#\{p\le x: p\nmid 2a, \ell_{-a}(p) \text{ not divisible by any } q^{e_q}\}.$$

Since both a^2 and $-a$ are positive, we can now compute $c_{a,f}$ by using the previous argument to estimate both right-hand side terms. We omit the details, mentioning only that (by a straightforward but laborious check) the density $c_{a,f}$ so obtained is positive in every case.

5. An exercise in heuristic reasoning

In this section, we propose an asymptotic formula for the number of $p\le x$ which divide some $a^{f(n)}-1$, where a and f are as in Theorem 1. For simplicity, we restrict ourselves to the case when $a>0$, and we assume that a is not a perfect power.

We adopt some notation from the previous section. Namely, we let $\mathscr{Q}$ be the set of primes q for which f does not have a zero modulo every power of q. For each $q\in\mathscr{Q}$, we let e_q be the minimal positive integer for which the congruence $f(n) \equiv 0 \pmod{q^{e_q}}$ is insoluble. Since $\mathscr{Q} \setminus \mathscr{P}$ is finite, we have that $e_q = 1$ for all but finitely many $q \in \mathscr{Q}$. Let

$$Q_1 := \prod_{\substack{q \mid [2,a_1] \\ q \in \mathscr{Q}}} q^{e_q}.$$

If $[2, a_1] \nmid Q_1$, then put $\nu = 1$; otherwise, define ν by (7).

Let χ denote the characteristic function of those natural numbers n divisible by no prime power q^{e_q}, with $q \in \mathscr{Q}$. Then χ is multiplicative. Moreover, p divides some $a^{f(n)} - 1$ precisely when $\chi(\ell(p)) = 1$. One can approximate the condition that $\chi(\ell(p)) = 1$ by the condition that $\ell(p)$ be divisible by no q^{e_q}, with q up to some fixed large parameter z. For *fixed* z, there is no difficulty in computing the relative density of primes satisfying this latter condition; indeed, the proof of Proposition 6 shows that this proportion is given by (8), where now $Q := \prod_{q \in \mathscr{Q} \cap [2,z]} q^{e_q}$. We now (unjustifiably) replace z with x to obtain the naive guess that

$$\begin{aligned}(9)\quad & \frac{1}{\pi(x)} \#\{p \leq x : \chi(\ell(p)) = 1\} \approx \\ & \left(1 + (\nu - 1)(-1)^{\omega(Q_1)} \prod_{q \mid Q_1} \frac{q}{q^{e_q+1} - q - q^{e_q-1}}\right) \prod_{q \in \mathscr{Q} \cap [2,x]} \left(1 - \frac{q^2}{q^{e_q}(q^2-1)}\right).\end{aligned}$$

Let us compare this prediction with what the same naive heuristic suggests for the total number of $n \leq x$ with $\chi(n) = 1$. Since $q^{e_q} \mid n$ with probability q^{-e_q}, our naive guess here is that

$$\frac{1}{x} \#\{n \leq x : \chi(n) = 1\} \approx \prod_{q \in \mathscr{Q} \cap [2,x]} \left(1 - \frac{1}{q^{e_q}}\right). \tag{10}$$

Dividing (9) by (10), we might conjecture that

$$\frac{\frac{1}{\pi(x)} \#\{p \leq x : \chi(\ell(p)) = 1\}}{\frac{1}{x} \#\{n \leq x : \chi(n) = 1\}} \to C_{a,f} \qquad (\text{as } x \to \infty), \tag{11}$$

where

$$C_{a,f} = \left(1 + (\nu - 1)(-1)^{\omega(Q_1)} \prod_{q \mid Q_1} \frac{q}{q^{e_q+1} - q - q^{e_q-1}}\right) \prod_{q \in \mathscr{Q}} \left(1 - \frac{1}{(q^2-1)(q^{e_q}-1)}\right). \tag{12}$$

As with $c_{a,f}$ in the last section, the definition of ν permits one to check in a straightforward way that $C_{a,f} > 0$.

To obtain our conjectured asymptotic formula, it remains to estimate the size of the denominator in (11), i.e., the number of $n \leq x$ for which $\chi(n) = 1$. This can be obtained from a theorem of Wirsing [21, Satz 1]. We state his result in a weaker form that suffices for our application.

Theorem D. *Let f be a multiplicative function satisfying $0 \le f(n) \le 1$ for all n. Assume that for some positive constant τ, one has $\sum_{p \le x} f(p) \sim \tau x/\log x$, as $x \to \infty$. Then*

$$\frac{1}{x}\sum_{n \le x} f(n) \sim \frac{1}{\log x}\frac{\mathrm{e}^{-\gamma\tau}}{\Gamma(\tau)}\prod_{p \le x}\left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right) \qquad (\text{as } x \to \infty).$$

Here γ is the Euler–Mascheroni constant and $\Gamma(z)$ is the classical Gamma function.

We take $f = \chi$ in Theorem D. By the Chebotarev density theorem (in the form of [15, Théorème 2], say), the hypothesis on $\sum_{p \le x} f(p)$ is satisfied with $\tau = 1 - r_f$. (Recall from the introduction that $1 - r_f > 0$.) Moreover, a short computation shows that

$$\prod_{p \le x}\left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right) = \prod_{p \le x}\left(1 - \frac{1}{p}\right)^{-1}\prod_{q \in \mathscr{Q} \cap [2,x]}\left(1 - \frac{1}{q^{e_q}}\right).$$

Invoking Mertens's theorem, we deduce that (as $x \to \infty$)

$$\frac{1}{x}\#\{n \le x : \chi(n) = 1\} \sim \frac{e^{r_f\gamma}}{\Gamma(1 - r_f)}\prod_{q \in \mathscr{Q} \cap [2,x]}\left(1 - \frac{1}{q^{e_q}}\right).$$

Comparing this with (11), and recalling that $\pi(x) \sim x/\log x$, we arrive at our conjecture:

Conjecture 7. *With the above notation and hypotheses, the number of primes $p \le x$ which divide $a^{f(n)} - 1$ for some n is*

$$\sim C_{a,f}\frac{\mathrm{e}^{r_f\gamma}}{\Gamma(1 - r_f)}\frac{x}{\log x}\prod_{q \in \mathscr{Q} \cap [2,x]}\left(1 - \frac{1}{q^{e_q}}\right) \qquad (\text{as } x \to \infty), \tag{13}$$

where $C_{a,f}$ is given by (12).

Remark. Lest the reader be misled, we should note that our heuristic does not depend on interpreting the symbol "≈" appearing in (9) and (10) as asymptotic equality. In fact, we expect that both naive predictions (9) and (10) are off by a constant factor; the hope is that this anomalous factor disappears upon dividing (9) by (10). More colloquially, we are hoping that two wrongs make a right!

In defense of this reasoning, we point out that an exactly analogous procedure leads to a number of widely accepted conjectures, including the quantitative form of the twin prime conjecture, the Murata–Pomerance conjecture on the number of $p \le x$ for which $\ell_2(p)$ is prime [12], and Motohashi's conjecture [11, Conjecture J*] on the number of $p \le x$ of the form $x^2 + y^2 + 1$, in the corrected form of Iwaniec [7].

Example. We give an example where the product appearing in (13) can be put in a more satisfactory form. Take $a = 2$ and $f(T) = T^2 + 1$. Then $\mathscr{Q}$ consists of 2 together with the primes $q \equiv 3 \pmod 4$; also, $e_q = 1$ for all $q \in \mathscr{Q}$ except $q = 2$, where $e_2 = 2$. We have $Q_1 = 4$, and so $\nu = 5/4$. From (12), we find that

$$C_{2,T^2+1} = \frac{7}{9}\prod_{q \equiv 3 \pmod 4}\left(1 - \frac{1}{(q^2 - 1)(q - 1)}\right).$$

Also, $r_f = \frac{1}{2}$, $\Gamma(1-r_f) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$, and by a theorem of Uchiyama [17],

$$\prod_{\substack{q \le x \\ q \equiv 3 \pmod 4}} \left(1 - \frac{1}{q}\right) \sim \mathrm{e}^{-\gamma/2} \sqrt{\frac{\pi}{2}} \left(\prod_{q \equiv 3 \pmod 4} \left(1 - \frac{1}{q^2}\right)^{1/2} \right) (\log x)^{-1/2}.$$

Thus, Conjecture 7 predicts that the number of $p \le x$ dividing some $2^{n^2+1} - 1$ is asymptotically

$$\frac{7}{12\sqrt{2}} \left(\prod_{q \equiv 3 \pmod 4} \left(1 - \frac{1}{q^2}\right)^{1/2} \left(1 - \frac{1}{(q^2-1)(q-1)}\right) \right) \frac{x}{(\log x)^{3/2}}.$$

An analogous simplification of the product appearing in (13) is possible whenever the splitting field of f has an abelian Galois group; see [20, 9].

6. Concluding remarks

As noted by Ballot and Luca, classical results on primitive prime divisors imply that for every choice of a and f, infinitely many primes p divide some $a^{f(n)} - 1$. But this argument gives only a very weak lower bound on the number of such $p \le x$. Can we do better?

Conjecture 7 is probably intractable at present. Even obtaining a lower bound of the form $\gg x/(\log x)^{1+r_f}$ seems difficult in general. It is more or less equivalent to asking for lower bounds of the expected order when one sieves the sequence $\{\ell(p)\}_{p \le x}$ by the set of primes $\mathscr{P}$ defined in (1). One may compare the situation with Hooley's GRH-conditional resolution of Artin's primitive root conjecture [5], which depends on sifting the corresponding sequence of indices $\{(p-1)/\ell(p)\}_{p \le x}$. We expect our problem to be at least as difficult as Hooley's. Indeed, as we saw in the proof of Theorem 1, under GRH the numbers $(p-1)/\ell(p)$ have only very small prime factors. This means that Hooley has only to sieve by a set of very small primes, which is quite convenient. We do not have this luxury.

Since (under GRH) the numbers $p-1$ and $\ell(p)$ have the same set of large prime factors, our problem is intimately related to the problem of sifting the set of shifted primes $p-1$ by a set like our $\mathscr{P}$. Here it seems very few lower bound results are known, apart from what can be derived from the half-dimensional sieve. To take a case that is favorable for us, consider the polynomial $f(T) = T^2 + 1$: From the half-dimensional sieve (as applied in [6]; cf. [2, p. 282, Theorem 14.8]), one obtains (unconditionally) $\gg x/(\log x)^{3/2}$ primes $p \le x$ for which $\frac{p-1}{2}$ is supported on primes $\equiv 1 \pmod 4$. For such primes, $\ell(p) \mid p-1 \mid n^2+1$ for some n, and so $p \mid a^{n^2+1} - 1$ (provided that $p \nmid a$). Since $r_f = \frac{1}{2}$, the lower bound agrees with the conjectured order of magnitude. Unfortunately, this unconditional proof appears not to generalize very far, not even to all pairs a and f with f quadratic. It would be interesting to know the extent to which extra hypotheses, like GRH, would allow us to extend the list of pairs a and f for which the conjecture can be proved.

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References

[1] Ballot, C.; Luca, F. Prime factors of $a^{f(n)} - 1$ with an irreducible polynomial $f(x)$, *New York J. Math.* **12** (2006) 39–45. MR 2217162 (2007b:11140), Zbl 1197.11127.

[2] Friedlander, J.; Iwaniec, H. Opera de cribro, American Mathematical Society Colloquium Publications, vol. 57, *American Mathematical Society, Providence, RI*, 2010. MR 2647984 (2011d:11227), Zbl pre05757681.

[3] Granville, A. Smooth numbers: computational number theory and beyond, *Algorithmic number theory: lattices, number fields, curves and cryptography*, Math. Sci. Res. Inst. Publ., vol. 44, *Cambridge Univ. Press, Cambridge*, 2008, pp. 267–323. MR 2467549 (2010g:11214), Zbl pre05532105.

[4] Halberstam, H.; Richert, H.-E. Sieve methods, London Mathematical Society Monographs, no. 4, *Academic Press, London-New York*, 1974. MR 0424730 (54 #12689), Zbl 0298.10026.

[5] Hooley, C. On Artin's conjecture, *J. Reine Angew. Math.* **225** (1967) 209–220. MR 0207630 (34 #7445), Zbl 0221.10048.

[6] Huxley, M. N.; Iwaniec, H. Bombieri's theorem in short intervals, *Mathematika* **22** (1975) 188–194. MR 0389790 (52 #10620), Zbl 0317.10048.

[7] Iwaniec, H. Primes of the type $\phi(x, y)+A$ where ϕ is a quadratic form, *Acta Arith.* **21** (1972) 203–234. MR 0304331 (46 #3466), Zbl 0215.35603.

[8] Lagarias, J. C.; Odlyzko, A. M. Effective versions of the Chebotarev density theorem, *Algebraic number fields: L-functions and Galois properties* (Proc. Sympos., Univ. Durham, Durham, 1975), *Academic Press, London*, 1977, pp. 409–464. MR 0447191 (56 #5506), Zbl 0362.12011.

[9] Languasco, A.; Zaccagnini, A. On the constant in the Mertens product for arithmetic progressions. I. Identities, *Funct. Approx. Comment. Math.* **42** (2010) 17–27. MR 2640766 (2011b:11127), Zbl 1206.11112.

[10] Moree, P. On primes p for which d divides $\mathrm{ord}_p(g)$, *Funct. Approx. Comment. Math.* **33** (2005) 85–95. MR 2274151 (2007j:11131), Zbl pre05135205.

[11] Motohashi, Y. On the distribution of prime numbers which are of the form $x^2 + y^2 + 1$, *Acta Arith.* **16** (1969/1970) 351–363. MR 0288086 (44 #5284), Zbl 0205.06801.

[12] Murata, L.; Pomerance, C. On the largest prime factor of a Mersenne number, *Number theory*, CRM Proc. Lecture Notes, vol. 36, *Amer. Math. Soc., Providence, RI*, 2004, pp. 209–218. MR 2076597 (2005i:11137), Zbl 1077.11003.

[13] Pappalardi, F., Square free values of the order function, *New York J. Math.* **9** (2003) 331–344. MR 2028173 (2004i:11116), Zbl 1066.11044.

[14] Ribenboim, P. Algebraic numbers, Pure and Applied Mathematics, no. 27, *Wiley-Interscience, New York-London-Sydney*, 1972. MR 0340212 (49 #4968), Zbl 0247.12002.

[15] Serre, J.-P. Quelques applications du théorème de densité de Chebotarev, *Inst. Hautes Études Sci. Publ. Math.* **54** (1981) 323–401. MR 0644559 (83k:12011), Zbl 0496.12011.

[16] Stevenhagen, P.; Lenstra, H. W., Jr. Chebotarëv and his density theorem, *Math. Intelligencer* **18** (1996), no. 2, 26–37. MR 1395088 (97e:11144), Zbl 0885.11005.

[17] Uchiyama, S. On some products involving primes, *Proc. Amer. Math. Soc.* **28** (1971) 629–630. MR 0277494 (43 #3227), Zbl 0212.07901.

[18] Wagstaff, S. S., Jr. Pseudoprimes and a generalization of Artin's conjecture, *Acta Arith.* **41** (1982) 141–150. MR 0674829 (83m:10004), Zbl 0496.10001.

[19] Wiertelak, K. On the density of some sets of primes. IV, *Acta Arith.* **43** (1984) 177–190. MR 0736730 (86e:11081), Zbl 0531.10049.

[20] Williams, K. S. Mertens' theorem for arithmetic progressions, *J. Number Theory* **6** (1974) 353–359. MR 0364137 (51 #392), Zbl 0286.10022.

[21] Wirsing, E. Das asymptotische Verhalten von Summen über multiplikative Funktionen, *Math. Ann.* **143** (1961) 75–102. MR 0131389 (24 #A1241), Zbl 0104.04201.

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