THE TRUTH ABOUT TORSION IN THE CM CASE

PETE L. CLARK AND PAUL POLLACK

ABSTRACT. Let $T_{\mathbf{CM}}(d)$ be the maximum size of the torsion subgroup of an elliptic curve with complex multiplication defined over a degree d number field. We show that there is an absolute, effective constant C such that $T_{\mathbf{CM}}(d) \leq Cd \log\log d$ for all $d \geq 3$.

For a commutative group G, we denote by $G[\text{tors}]$ the torsion subgroup of G.

1. INTRODUCTION

The aim of this note is to prove the following result.

Theorem 1. *There is an absolute, effective constant C such that for all number fields F of degree $d \geq 3$ and all elliptic curves $E_{/F}$ with complex multiplication,*

$$\#E(F)[\text{tors}] \leq Cd \log\log d.$$

It is natural to compare this result with the following one.

Theorem 2 (Hindry–Silverman [HS99])**.** *For all number fields F of degree $d \geq 2$ and all elliptic curves $E_{/F}$ with j-invariant $j(E) \in \mathcal{O}_F$, we have*

$$\#E(F)[\text{tors}] \leq 1977408 d \log d.$$

Every CM elliptic curve $E_{/F}$ has $j(E) \in \mathcal{O}_F$, but only finitely many $j \in \mathcal{O}_F$ are j-invariants of CM elliptic curves $E_{/F}$. Thus Theorem 1 has a significantly stronger hypothesis and a slightly stronger conclusion than Theorem 2. But the improvement of $\log\log d$ over $\log d$ is interesting in view of the following result.

Theorem 3 (Breuer [Br10])**.** *Let $E_{/F}$ be an elliptic curve over a number field. There exists a constant $c(E, F) > 0$, integers $3 \leq d_1 < d_2 < \ldots < d_n < \ldots$ and number fields $F_n \supset F$ with $[F_n : F] = d_n$ such that for all $n \in \mathbb{Z}^+$ we have*

$$\#E(F_n)[\text{tors}] \geq \begin{cases} c(E, F) d_n \log\log d_n & \text{if } E \text{ has CM,} \\ c(E, F)\sqrt{d_n \log\log d_n} & \text{otherwise.} \end{cases}$$

Let $T_{\mathbf{CM}}(d)$ be the maximum size of the torsion subgroup of a CM elliptic curve over a degree d number field. Then Theorems 1 and 3 combine to tell us that the *upper order* of $T_{\mathbf{CM}}(d)$ is $d \log\log d$:

$$0 < \limsup_{d \to \infty} \frac{T_{\mathbf{CM}}(d)}{d \log\log d} < \infty.$$

To our knowledge, this is the first instance of an upper order result for torsion points on a class of abelian varieties over number fields of varying degree.

Define $T(d)$ as for $T_{\mathbf{CM}}(d)$ but replacing "CM elliptic curve" with "elliptic curve", and define $T_{\neg\mathbf{CM}}(d)$ as for $T_{\mathbf{CM}}(d)$ but replacing "CM elliptic curve" with "elliptic curve *without* CM". Hindry and Silverman ask whether $T_{\neg\mathbf{CM}}(d)$ has upper order $\sqrt{d \log\log d}$. If so, the upper order of $T(d)$ would be $d \log\log d$ [CCRS13, Conjecture 1].

2. Proof of the Main Theorem

2.1. Torsion Points and Ray Class Containment

Let K be a number field. Let $\mathcal{O}_K$ be the ring of integers of K, Δ_K the discriminant of K, w_K the number of roots of unity in K and h_K the class number of K. By an "ideal of $\mathcal{O}_K$" we shall always mean a nonzero ideal. For an ideal $\mathfrak{a}$ of $\mathcal{O}_K$, we write $K^{(\mathfrak{a})}$ for the $\mathfrak{a}$-ray class field of K. We also put $|\mathfrak{a}| = \#\mathcal{O}_K/\mathfrak{a}$ and

$$\varphi_K(\mathfrak{a}) = \#(\mathcal{O}_K/\mathfrak{a})^\times = |\mathfrak{a}| \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 - \frac{1}{|\mathfrak{p}|}\right).$$

An elliptic curve E defined over a field of characteristic 0 has *complex multiplication (CM)* if $\operatorname{End} E \supsetneq \mathbb{Z}$; then $\operatorname{End} E$ is an order in an imaginary quadratic field. We say E has $\mathcal{O}$-CM if $\operatorname{End} E \cong \mathcal{O}$ and K-CM if $\operatorname{End} E$ is an order in K.

Lemma 4. *Let K be an imaginary quadratic field and $\mathfrak{a}$ an ideal of $\mathcal{O}_K$. Then*

$$\frac{h_K \varphi_K(\mathfrak{a})}{6} \leq \frac{h_K \varphi_K(\mathfrak{a})}{w_K} \leq [K^{(\mathfrak{a})} : K] \leq h_K \varphi_K(\mathfrak{a}).$$

Proof. This follows from [Co00, Corollary 3.2.4]. $\square$

Theorem 5. *Let K be an imaginary quadratic field, $F \supset K$ a number field, $E_{/F}$ a K-CM elliptic curve and $N \in \mathbb{Z}^+$. If $(\mathbb{Z}/N\mathbb{Z})^2 \hookrightarrow E(F)$, then $F \supset K^{(N\mathcal{O}_K)}$.*

Proof. The result is part of classical CM theory when $\operatorname{End} E = \mathcal{O}_K$ is the maximal order in K [Si94, II.5.6]. We shall reduce to that case. There is an $\mathcal{O}_K$-CM elliptic curve $E'_{/F}$ and a canonical F-rational isogeny $\iota \colon E \to E'$ [CCRS13, Prop. 25]. There is a field embedding $F \hookrightarrow \mathbb{C}$ such that the base change of ι to $\mathbb{C}$ is, up to isomorphisms on the source and target, given by $\mathbb{C}/\mathcal{O} \to \mathbb{C}/\mathcal{O}_K$. If we put

$$P = 1/N + \mathcal{O} \in E[N], \quad P' = 1/N + \mathcal{O}_K \in E'[N],$$

then $\iota(P) = P'$ and P' generates $E'[N]$ as an $\mathcal{O}_K$-module. By assumption $P \in E(F)$, so $\iota(P) = P' \in E'(F)$. It follows that $(\mathbb{Z}/N\mathbb{Z})^2 \hookrightarrow E'(F)[\text{tors}]$. $\square$

Remark 6. *In fact one can show — e.g., using adelic methods — that for any K-CM elliptic curve E defined over $\mathbb{C}$, the field obtained by adjoining to $K(j(E))$ the values of the Weber function at the N-torsion points of E contains $K^{(N\mathcal{O}_K)}$.*

2.2. Squaring the Torsion Subgroup of a CM Elliptic Curve

Theorem 7. *Let K be an imaginary quadratic field, let $F \supset K$ a field extension, and let $E_{/F}$ be a K-CM elliptic curve. Suppose that for positive integers a and b we have an injection $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/ab\mathbb{Z} \hookrightarrow E(F)$. Then $[F(E[ab]) : F] \leq b$.*

Proof. **Step 1**: Let $\mathcal{O} = \operatorname{End} E$. For $N \in \mathbb{Z}^+$, let $C_N = (\mathcal{O}/N\mathcal{O})^\times$. Let $E[N] = E[N](\overline{F})$. As an $\mathcal{O}/N\mathcal{O}$-module, $E[N]$ is free of rank 1. Let $\mathfrak{g}_F = \operatorname{Aut}(\overline{F}/F)$, and let $\rho_N \colon \mathfrak{g}_F \longrightarrow \operatorname{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be the mod N Galois representation associated to $E_{/F}$. Because E has $\mathcal{O}$-CM and $F \supset K$, we have

$$\rho_N \colon \mathfrak{g}_F \longrightarrow \operatorname{Aut}_{\mathcal{O}} E[N] \cong \operatorname{GL}_1(\mathcal{O}/N\mathcal{O}) \cong (\mathcal{O}/N\mathcal{O})^\times = C_N.$$

Let Δ be the discriminant of $\mathcal{O}$. Then $e_1 = 1$, $e_2 = \frac{\Delta+\sqrt{\Delta}}{2}$ is a $\mathbb{Z}$-basis for $\mathcal{O}$. The induced ring embedding $\mathcal{O} \hookrightarrow M_2(\mathbb{Z})$ is given by $\alpha e_1 + \beta e_2 \mapsto \begin{bmatrix} \alpha & \frac{\beta\Delta - \beta\Delta^2}{4} \\ \beta & \alpha + \beta\Delta \end{bmatrix}$. So

$$C_N = \left\{ \begin{bmatrix} \alpha & \frac{\beta\Delta - \beta\Delta^2}{4} \\ \beta & \alpha + \beta\Delta \end{bmatrix} \mid \alpha, \beta \in \mathbb{Z}/N\mathbb{Z}, \text{ and} \right.$$

$$\left. \alpha^2 + \Delta\alpha\beta + \left(\frac{\Delta^2 - \Delta}{4} \right) \beta^2 \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

From this we easily deduce the following useful facts:

(i) C_N contains the homotheties $\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \mid \alpha \in (\mathbb{Z}/N\mathbb{Z})^\times \}$.

(ii) For all primes p and all $A, B \geq 1$, the natural reduction map $C_{p^{A+B}} \to C_{p^A}$ is surjective and its kernel has size p^{2B}.

Step 2: Primary decomposition reduces us to the case $a = p^A$, $b = p^B$ with $A \geq 0$ and $B \geq 1$. By induction it suffices to treat the case $B = 1$: i.e., we assume $E(F)$ contains full p^A-torsion and a point of order p^{A+1} and show $[F(E[p^{A+1}]) : F] \leq p$.

Case $A = 0$:

- If $\left(\frac{\Delta}{p} \right) = 1$, then C_p is conjugate to $\left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{F}_p^\times \right\}$. If $\alpha \neq 1$ (resp. $\beta \neq 1$) the only fixed points $(x, y) \in \mathbb{F}_p^2$ of $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ have $x = 0$ (resp. $y = 0$). Because $E(F)$ contains a point of order p we must either have $\alpha = 1$ for all $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in \rho_p(\mathfrak{g}_F)$ or $\beta = 1$ for all $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in \rho_p(\mathfrak{g}_F)$. Either way, $\#\rho_p(\mathfrak{g}_F) \mid p - 1$.
- If $\left(\frac{\Delta}{p} \right) = -1$, then C_p acts simply transitively on $E[p] \setminus \{0\}$, so if we have one F-rational point of order p then $E[p] \subset E(F)$, so $\#\rho_p(\mathfrak{g}_F) = 1$.
- If $\left(\frac{\Delta}{p} \right) = 0$, then C_p is conjugate to $\left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \mid \alpha \in \mathbb{F}_p^\times, \ \beta \in \mathbb{F}_p \right\}$ [BCS15, §4.2]. Since $E(F)$ has a point of order p, every element of $\rho_p(\mathfrak{g}_F)$ has 1 as an eigenvalue and thus $\rho_p(\mathfrak{g}_F) \subset \{ \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \mid \beta \in \mathbb{F}_p \}$, so has order dividing p.

Case $A \geq 1$: By (ii), $\mathcal{K} = \ker C_{p^{A+1}} \to C_{p^A}$ has size p^2. Since $(\mathbb{Z}/p^A\mathbb{Z})^2 \hookrightarrow E(F)$, we have $\rho_{p^{A+1}}(\mathfrak{g}_F) \subset \mathcal{K}$. Since $E(F)$ has a point of order p^{A+1}, by (i) the homothety $\begin{bmatrix} 1+p^A & 0 \\ 0 & 1+p^A \end{bmatrix}$ lies in $\mathcal{K} \setminus \rho_{p^{A+1}}(\mathfrak{g}_F)$. Therefore $\rho_{p^{A+1}}(\mathfrak{g}_F) \subsetneq \mathcal{K}$, so $\#\rho_{p^{A+1}}(\mathfrak{g}_F) \mid p$. $\square$

2.3. Uniform Bound for Euler's Function in Imaginary Quadratic Fields

Let $\mathfrak{a}$ be an ideal in an imaginary quadratic field K. To apply the results of §2.1, we require a lower bound on $\frac{\varphi_K(\mathfrak{a})}{|\mathfrak{a}|}$. For *fixed* K, it is straightforward to adapt a classical argument of Landau (see the proof of [HW08, Theorem 328, p. 352]). Replacing Landau's use of Mertens' Theorem with Rosen's number field analogue [Ro99], one obtains the following result: let γ denote the Euler–Mascheroni constant, and let $\chi(\cdot) = \left(\frac{\Delta_K}{\cdot} \right)$ be the quadratic Dirichlet character associated to K. Then

$$\liminf_{|\mathfrak{a}| \to \infty} \frac{\varphi_K(\mathfrak{a})}{|\mathfrak{a}|/\log\log|\mathfrak{a}|} = e^{-\gamma} \cdot L(1, \chi)^{-1}.$$

Unfortunately, this result is not sufficient for our purposes. There are two sources of difficulty. First, the right-hand side depends on K, and can in fact be arbitrarily small (see [BCE50, (4′)]). Second, the statement only addresses limiting behavior as $|\mathfrak{a}| \to \infty$, and we need a result with no such restriction on $|\mathfrak{a}|$. However, looking back at Lemma 4 we see that a lower bound on $h_K \frac{\varphi_K(\mathfrak{a})}{|\mathfrak{a}|}$ would suffice. The factor of h_K allows us to prove a totally uniform lower bound.

Theorem 8. *There is a positive, effective absolute constant C such that for all imaginary quadratic fields K and all nonzero ideals $\mathfrak{a}$ of $\mathcal{O}_K$ with $|\mathfrak{a}| \geq 3$, we have*

$$\varphi_K(\mathfrak{a}) \geq \frac{C}{h_K} \cdot \frac{|\mathfrak{a}|}{\log\log|\mathfrak{a}|}.$$

Lemma 9. *For a fundamental quadratic discriminant $\Delta < 0$ let $K = \mathbb{Q}(\sqrt{\Delta})$, and let $\chi(\cdot) = \left(\frac{\Delta}{\cdot}\right)$. There is an effective constant $C > 0$ such that for all $x \geq 2$,*

$$(1) \qquad \prod_{p \leq x}\left(1 - \frac{\chi(p)}{p}\right) \geq \frac{C}{h_K}.$$

Proof. By the quadratic class number formula, $h_K \asymp L(1,\chi)\sqrt{|\Delta|}$ [Da00, eq. (15), p. 49]. Writing $L(1,\chi) = \prod_p(1 - \chi(p)/p)^{-1}$ and rearranging, we see (1) holds iff

$$(2) \qquad \prod_{p > x}\left(1 - \frac{\chi(p)}{p}\right) \ll \sqrt{|\Delta|},$$

with an effective and absolute implied constant. By Mertens' Theorem [HW08, Theorem 429, p. 466], the factors on the left-hand side of (2) indexed by $p \leq \exp(\sqrt{|\Delta|})$ make a contribution of $O(\sqrt{|\Delta|})$. Put $y = \max\{x, \exp(\sqrt{|\Delta|})\}$; it suffices to show that $\prod_{p > y}(1 - \chi(p)/p) \ll 1$. Taking logarithms, this will follow if we prove that $\sum_{p > y}\chi(p)/p = O(1)$. For $t \geq \exp(\sqrt{|\Delta|})$, the explicit formula gives $S(t) := \sum_{p \leq t}\chi(p)\log p = -t^\beta/\beta + O(t/\log t)$, where the main term is present only if $L(s,\chi)$ has a Siegel zero β. (C.f. [Da00, eq. (8), p. 123].) We will assume the Siegel zero exists; otherwise the argument is similar but simpler. By partial summation,

$$\sum_{p > y}\frac{\chi(p)}{p} = -\frac{S(y)}{y\log y} + \int_y^\infty \frac{S(t)}{t^2(\log t)^2}(1 + \log t)\,dt$$

$$\ll 1 + \int_y^\infty \frac{t^\beta}{t^2\log t}\,dt.$$

Haneke, Goldfeld–Schinzel, and Pintz have each shown that $\beta \leq 1 - c/\sqrt{|\Delta|}$, where the constant $c > 0$ is absolute and effective [Ha73, GS75, Pi76]. Using this to bound t^β, and keeping in mind that $y \geq \exp(\sqrt{|\Delta|})$, we see that the final integral is at most

$$\int_{\exp(\sqrt{|\Delta|})}^\infty \frac{\exp(-c\log t/\sqrt{|\Delta|})}{t\log t}\,dt.$$

A change of variables transforms the integral into $\int_1^\infty \exp(-cu)u^{-1}\,du$, which converges. Assembling our estimates completes the proof. $\square$

Proof of Theorem 8. Write $\varphi_K(\mathfrak{a}) = |\mathfrak{a}|\prod_{\mathfrak{p}\mid\mathfrak{a}}(1 - 1/|\mathfrak{p}|)$, and notice that the factors are increasing in $|\mathfrak{p}|$. So if $z \geq 2$ is such that $\prod_{|\mathfrak{p}| \leq z}|\mathfrak{p}| \geq |\mathfrak{a}|$, then

$$(3) \qquad \frac{\varphi_K(\mathfrak{a})}{|\mathfrak{a}|} \geq \prod_{|\mathfrak{p}| \leq z}\left(1 - \frac{1}{|\mathfrak{p}|}\right).$$

We first establish a lower bound on the right-hand side, as a function of z, and then we prove the theorem by making a convenient choice of z. We partition the prime ideals

with $|\mathfrak{p}| \leq z$ according to the splitting behavior of the rational prime p lying below $\mathfrak{p}$. Noting that $p \leq |\mathfrak{p}|$, Mertens' Theorem and Lemma 9 yield

$$\prod_{|\mathfrak{p}| \leq z} \left(1 - \frac{1}{|\mathfrak{p}|}\right) \geq \prod_{p \leq z} \left(1 - \frac{1}{p}\right)\left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p}\right)$$

$$\text{(4)} \qquad\qquad \gg (\log z)^{-1} \prod_{p \leq z} \left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p}\right) \gg (\log z)^{-1} \cdot h_K^{-1}.$$

With C' a large absolute constant to be described momentarily, we set

$$\text{(5)} \qquad\qquad z = (C' \log |\mathfrak{a}|)^2.$$

We must check that $\prod_{|\mathfrak{p}| \leq z} |\mathfrak{p}| \geq |\mathfrak{a}|$. The Prime Number Theorem implies

$$\prod_{|\mathfrak{p}| \leq z} |\mathfrak{p}| \geq \prod_{p \leq z^{1/2}} p \geq \prod_{p \leq C' \log |\mathfrak{a}|} p \geq |\mathfrak{a}|,$$

provided that C' was chosen appropriately. Combining (3), (4), and (5) gives

$$\varphi_K(\mathfrak{a}) \gg |\mathfrak{a}| \cdot (\log z)^{-1} \cdot h_K^{-1} \gg h_K^{-1} \cdot |\mathfrak{a}| \cdot \log(\log(|\mathfrak{a}|))^{-1}. \qquad \square$$

2.4. Proof of Theorem 1

Let F be a number field of degree $d \geq 3$, and let $E_{/F}$ be a K-CM elliptic curve. We may assume $\#E(F)[\text{tors}] \geq 3$. We have $E(FK)[\text{tors}] \cong \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/ab\mathbb{Z}$ for positive integers a and b. Theorem 5 gives $FK \supset K^{(a\mathcal{O}_K)}$. Along with Lemma 4 we get

$$2d \geq [FK : \mathbb{Q}] \geq [K^{(a\mathcal{O}_K)} : \mathbb{Q}] \geq \frac{h_K \varphi_K(a\mathcal{O}_K)}{3}.$$

By Theorem 7, there is an extension L/FK with $(\mathbb{Z}/ab\mathbb{Z})^2 \hookrightarrow E(L)$ and $[L : FK] \leq b$. Applying Theorem 5 and Lemma 4 as above we get $L \supset K^{(ab\mathcal{O}_K)}$ and

$$[L : \mathbb{Q}] \geq [K^{(ab\mathcal{O}_K)} : \mathbb{Q}] \geq \frac{h_K \varphi_K(ab\mathcal{O}_K)}{3},$$

so

$$\text{(6)} \qquad d = [F : \mathbb{Q}] \geq \frac{[FK : \mathbb{Q}]}{2} = \frac{[L : \mathbb{Q}]}{2[L : FK]} \geq \frac{[L : \mathbb{Q}]}{2b} \geq \frac{h_K \varphi_K(ab\mathcal{O}_K)}{6b}.$$

Multiplying (6) through by $(ab)^2 = |ab\mathcal{O}_K|$ and rearranging, we get

$$\text{(7)} \qquad\qquad \#E(FK)[\text{tors}] = a^2 b \leq 6 \frac{d}{h_K} \frac{|ab\mathcal{O}_K|}{\varphi_K(ab\mathcal{O}_K)}.$$

By Theorem 8 we have

$$\text{(8)} \qquad \frac{|ab\mathcal{O}_K|}{\varphi_K(ab\mathcal{O}_K)} \ll h_K \log\log|ab\mathcal{O}_K| \leq h_K \log\log(a^2 b)^2 \ll h_K \log\log \#E(FK)[\text{tors}].$$

Combining (7) and (8) gives

$$\#E(FK)[\text{tors}] \ll d \log\log \#E(FK)[\text{tors}]$$

and thus

$$\#E(F)[\text{tors}] \leq \#E(FK)[\text{tors}] \ll d \log\log d.$$

2.5. **Related Work**

Let E be a K-CM elliptic curve defined over a number field F, and let $P \in E(F)[\text{tors}]$. Silverberg showed [Si92, Corollary 6.1] that if $F \supset K$ then $\varphi(\#\langle P \rangle) \leq 3[F : \mathbb{Q}]$. It follows that if $F \not\supset K$ then $\varphi(\#\langle P \rangle) \leq 6[F : \mathbb{Q}]$. Later Aoki showed [Ao95, Proposition 8.1] that if $F \not\supset K$ then $\varphi(\#\langle P \rangle) \leq 2[F : \mathbb{Q}]$. Both Silverberg's and Aoki's bounds are *the real truth*: there are points of order 6 when $F = \mathbb{Q}$ and of order 7 when $F = K = \mathbb{Q}(\sqrt{-3})$.

These results imply a $O(d \log \log d)$ bound on the *exponent* of $E(F)[\text{tors}]$ and thus that $\#E(F)[\text{tors}] = O((d \log \log d)^2)$, which was later superceded by Theorem 2. When $F \not\supset K$, then $E(F)[\text{tors}]$ has a cyclic subgroup of index at most 2. Thus the work of Silverberg and Aoki yields Theorem 1 when $F \not\supset K$, in fact in the more explicit form

$$\#E(F)[\text{tors}] \leq (4e^\gamma + o(1))d \log \log d, \quad \text{as } d \to \infty.$$

References

[Ao95]   N. Aoki, *Torsion points on abelian varieties with complex multiplication*. Algebraic cycles and related topics (Kitasakado, 1994), 1–22, World Sci. Publ., River Edge, NJ, 1995.

[BCE50] P.T. Bateman, S. Chowla and P. Erdős, *Remarks on the size of $L(1, \chi)$*. Publ. Math. Debrecen 1 (1950), 165–182.

[BCS15] A. Bourdon, P.L. Clark and J. Stankewicz, *Torsion points on CM elliptic curves over real number fields*, submitted for publication.

[Br10]   F. Breuer, *Torsion bounds for elliptic curves and Drinfeld modules*. J. Number Theory 130 (2010), 1241–1250.

[CCRS13] P.L. Clark, B. Cook and J. Stankewicz, *Torsion points on elliptic curves with complex multiplication (with an appendix by Alex Rice)*. International Journal of Number Theory 9 (2013), 447–479.

[Co00]   H. Cohen, *Advanced Topics in Computational Number Theory*. Graduate Texts in Mathematics 193, Springer-Verlag, 2000.

[Da00]   H. Davenport, *Multiplicative Number Theory*. Third edition. Graduate Texts in Mathematics, 74. Springer-Verlag, New York, 2000.

[Ha73]   W. Haneke, *Über die reellen Nullstellen der Dirichletschen L-Reihen*. Acta Arith. 22 (1973), 391–421. Corrigendum in 31 (1976), 99–100.

[HS99]   M. Hindry and J. Silverman, *Sur le nombre de points de torsion rationnels sur une courbe elliptique*. C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), no. 2, 97–100.

[HW08]   G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*. Sixth edition. Oxford University Press, Oxford, 2008.

[GS75]   D.M. Goldfeld and A. Schinzel, *On Siegel's zero*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), 571–583.

[Pi76]   J. Pintz, *Elementary methods in the theory of L-functions. II. On the greatest real zero of a real L-function*. Acta Arith. 31 (1976), 273–289.

[Ro99]   M. Rosen, *A generalization of Mertens' Theorem*. J. Ramanujan Math. Soc. 14 (1999), 1–19.

[Si92]   A. Silverberg, *Points of finite order on abelian varieties*. In *p-adic methods in number theory and algebraic geometry*, 175–193, Contemp. Math. 133, Amer. Math. Soc., Providence, RI, 1992.

[Si94]   J. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 151, Springer-Verlag, 1994.