Counting perfect numbers

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Beginning at the beginning

Recall that a *perfect number* is a natural number N satisfying

$$\sigma(N) = 2N, \quad \text{where} \quad \sigma(N) = \sum_{d|N} d$$

is the usual sum-of-divisors function.

Let $V(x)$ be the number of perfect $N \leq x$.
Write $V(x) = V_0(x) + V_1(x)$, where $V_0(x)$ is the number of even perfect numbers $\leq x$, and $V_1(x)$ is the number of odd perfect numbers $\leq x$.

If N is even perfect, then (Euler)

$$N = 2^{n-1}(2^n - 1)$$

where $2^n - 1$ is prime, and conversely (Euclid).

So trivially, $V_0(x) \ll \log x$.

Conjecture

As $x \to \infty$, we have

$$V_0(x) \sim \frac{e^\gamma}{\log 2} \log\log x.$$

Conjecture

There are no odd perfect numbers.

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Hornfeck & Wirsing, 1957	$V(x) = O(x^{\epsilon})$
Wirsing, 1959	$V(x) \leq x^{W/\log \log x}$

Euler's structure theorem

Theorem

Let N be an odd perfect number. Then N has the form $p^e M^2$, where $p \equiv e \equiv 1 \pmod 4$ and $\gcd(p, M) = 1$.

Proof (sketch).

Since $\sigma(N) = 2N$, we have that $2 \mid \sigma(N)$ but $2^2 \nmid \sigma(N)$.
If $N = \prod p^{e_p}$, then

$$\sigma(N) = \prod \sigma(p^{e_p}) = \prod_p (1 + p + \cdots + p^{e_p}).$$

All but one factor here must be odd, and that factor must be divisible by 2 but not 2^2.

Hornfeck's bound

Again we estimate the number of odd perfect $N \leq x$. This time we show the number is up to x is bounded by

$$x^{1/2}.$$

Write

$$N = p^e M^2.$$

Clearly $M^2 < N \leq x$, so $M \leq x^{1/2}$.
We will show that M determines p^e, and so also N.

We have

$$2p^e M^2 = 2N = \sigma(N) = \sigma(p^e)\sigma(M^2)$$

and hence

$$\frac{\sigma(p^e)}{p^e} = 2\frac{M^2}{\sigma(M^2)}.$$

Left-hand fraction is in lowest terms. So p^e is the denominator when $2M^2/\sigma(M^2)$ is put in lowest terms. This depends only on M.

Wirsing's method

Let N be a perfect number.
Let $B > 1$ be a *unitary divisor* of N, so that

$$N = AB \quad \text{with} \quad \gcd(A, B) = 1.$$

Unapologetically vague goal

Show that N is determined by B and "a little bit more".

Example

If $N = p^e M^2$ is odd perfect, and we take $B = M^2$, then B by itself determines N.

The Wirsing algorithm

We now describe an algorithm which, given a perfect number N and a unitary divisor $B > 1$ of N, generates a finite (possibly empty) *exponent sequence* $e_0, ..., e_{l-1}$ of positive integers.

Moreover, there is a dual algorithm to reconstruct N from the pair $(B, \text{exponent sequence})$. In fact,

$$N = (p_0^{e_0} p_1^{e_1} \dots p_{l-1}^{e_{l-1}})B$$

for primes $p_0, \dots, p_{l-1}$ which are algorithmically determined by B and the exponent sequence.

The Wirsing algorithm

Given: N perfect, and $B > 1$ a unitary divisor of N.

Write $N = AB$, so that $\gcd(A, B) = 1$.
If $A = 1$, output the empty sequence and terminate.
Otherwise we have

$$\sigma(N) = \sigma(A)\sigma(B) = 2AB$$

and

$$1 < \frac{\sigma(A)}{A} = \frac{2B}{\sigma(B)} < 2.$$

So $2B/\sigma(B)$ is *not* an integer.

If $2B/\sigma(B)$ is not an integer, then let p_0 be the least prime dividing $\sigma(B)$ to a higher power than that to which it divides $2B$. Then $p_0 \mid A$. Note that p_0 is entirely determined by B.

Suppose $p_0^{e_0} \parallel A$. Then

$$N = AB = (A/p_0^{e_0})(Bp_0^{e_0}) = A_1 B_1.$$

Now $B_1 > 1$ is a unitary divisor of N. So either $A_1 = 1$, or we find that $2B_1/\sigma(B_1)$ is not an integer.

In the former case, output e_0 as the exponent sequence and quit. Otherwise, $2B_1/\sigma(B_1)$ is not an integer.

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Suppose $p_1^{e_1} \parallel A_1$. Then

$$N = A_1 B_1 = (A_1/p_1^{e_1})(B_1 p_1^{e_1}) = A_2 B_2.$$

Now $B_2 > 1$ is a unitary divisor of N. So either $A_2 = 1$, or we find that $2B_2/\sigma(B_2)$ is not an integer.

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In the former case, output e_0, e_1 as the exponent sequence and quit. Otherwise, $2B_2/\sigma(B_2)$ is not an integer.

We could keep going but you get the idea. This algorithm terminates!

Can we recover N?

This process eventually terminates with $A_l = 1$: Then

$$N = A_l B_l = B_l = B p_0^{e_0} \cdots p_{l-1}^{e_{l-1}}.$$

Here the prime p_i is determined by B and the exponents $e_0, e_1, \ldots, e_{i-1}$. So N can be completely reconstructed by knowledge of B and the exponent sequence e_0, ..., e_{l-1}.

Note that if we wrote our original factorization as $N = AB$, then

$$A = p_0^{e_0} \cdots p_{l-1}^{e_{l-1}}.$$

Application

We will prove the following theorem:

Theorem (P.)

Let $k \geq 2$. Suppose $x > e^{12}$. The number of odd perfect $N \leq x$ with $\leq k$ distinct prime factors is bounded by $(\log x)^{2k}$.

Let $N \leq x$ be odd perfect with $\leq k$ distinct prime factors, and write $N = AB$, where

$$p \mid A \Longrightarrow p > 2k$$

and

$$p \mid B \Longrightarrow p \leq 2k.$$

Notice that $B > 1$. Otherwise $N = A$. But

$$\begin{aligned} \frac{A}{\sigma(A)} &= \prod_{p^{v_p} \| A} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{v_p}}\right)^{-1} \\ &\geq \prod_{p \mid A} \left(1 - \frac{1}{p}\right) \geq 1 - \sum_{p \mid A} \frac{1}{p} \geq 1 - \frac{k}{2k+1} > \frac{1}{2}, \end{aligned}$$

so A is not perfect.

So apply the Wirsing algorithm to each pair (N, B) where N ranges over odd perfects $\leq x$ with at most k prime factors, and B is the $(2k)$-smooth part of N. Each time we get an exponent sequence $e_0, e_1, \ldots, e_{l-1}$.

Moreover, B and the sequence $e_0, e_1, e_2, \ldots$ determines N.

To count the N, we count the possible values of the pair $(B, \text{exponent sequence})$.

Counting Bs

Recall that B has the form $\prod_{3 \le p \le 2k} p^{v_p}$. For each $3 \le p \le 2k$, we have

$$3^{v_p} \le p^{v_p} \le B \le N \le x.$$

So $0 \le v_p \le \log x / \log 3$.

The number of odd primes $p \le 2k$ is smaller than k. So the number of choices for B is bounded by

$$(1 + \log x / \log 3)^k \le (\log x)^k.$$

Here we use that $x > e^{12}$.

Counting exponent sequences

How many choices are there for the exponent sequence $e_0, e_1, e_2, \ldots$?
At the end of the Wirsing process, we have a factorization of the form

$$A = p_0^{e_0} \cdots p_{l-1}^{e_{l-1}}.$$

Since $A \leq x$ and each odd prime divisor of A is $\geq 2k+1 \geq 5$, we have

$$5^{e_i} \leq p_i^{e_i} \leq A \leq x.$$

So $1 \leq e_i \leq \log x / \log 5$.
Moreover, the number of terms in the sequence $e_0, e_1, \ldots$ is $< k$.
So the number of possibilities for $e_0, e_1, e_2, \ldots$ is at most

$$k(\log x / \log 5)^k \leq (\log x)^k.$$

Application to Dickson's theorem

Theorem (Dickson)

For each fixed k, there are finitely many odd perfect numbers with at most k distinct prime factors.

So in place of our bound of $(\log x)^{2k}$, Dickson says we can take

$$C(k)$$

for large x.

Theorem (P.)

The number of odd perfect N with at most k distinct prime factors is smaller than

$$2^{(2k)^2}.$$

Proof.

By Heath-Brown et al., $N < 2^{2^{2k}}$.
Use the previous theorem to count the number of odd perfects $\leq x := 2^{2^{2k}}$ with k prime factors.
We get $(\log x)^{2k} < (2^{2k})^{2k} = 2^{(2k)^2}$.

Wirsing's bound for $V(x)$

Idea: For each perfect number $N \leq x$, write

$$N = AB,$$

where

$$p \mid A \Longrightarrow p > \log x,$$

$$p \mid B \Longrightarrow p \leq \log x.$$

Then

$$\frac{A}{\sigma(A)} > \prod_{p \mid A} (1 - 1/p) > 1 - \frac{1}{\log x} \sum_{p \mid A} 1.$$

Since each $p \mid A$ satisfies $p > \log x$, the number of primes p dividing A is $\leq \log x / \log\log x$. Hence $A/\sigma(A) > 1 - 1/\log\log x > 1/2$ for large x. So $B > 1$ and we get an exponent sequence $e_0, e_1, \ldots$.

Bounding the number of Bs, take 2

Let $\Psi(x, y)$ be the number of $n \leq x$ all of whose prime divisors are $\leq y$. Then each B is $(\log x)$-smooth.

Theorem (Erdős)

We have $\Psi(x, \log x) = x^{(1+o(1)) \log 4 / \log\log x}$.

It is easy to give an elementary proof that

$$\Psi(x, \log x) \leq x^{W_0 / \log\log x}$$

for some constant W_0, which is all we need for Wirsing's theorem.

Bounding the number of exponent sequences

This time we have

$$A = p_0^{e_0} p_1^{e_1} p_2^{e_2} \cdots ,$$

and

$$A \geq (\log x)^{e_0 + e_1 + \cdots}.$$

Since $A \leq x$, we have

$$e_0 + e_1 + \cdots \leq \log x / \log \log x.$$

Lemma

Let M be a positive integer. The number of sequences of positive integers $e_0, e_1, e_2, \ldots$ with $e_0 + e_1 + \cdots \leq M$ is precisely 2^M.

As a consequence, the number of possible exponent sequences is

$$\leq 2^{\lfloor \log x / \log \log x \rfloor} \leq 2^{\log x / \log \log x} = x^{\log 2 / \log \log x}.$$

Putting it together, we find that the number of perfect $N \le x$ is bounded by

$$x^{(\log 4+o(1))/\log\log x} x^{\log 2/\log\log x} = x^{(3\log 2+o(1))/\log\log x}.$$

So for any $W > 3\log 2$, we have

$$V(x) < x^{W/\log\log x}$$

for all large enough x.

Vistas

Question

How "far away" from integrality is $\frac{\sigma(N)}{N}$?

Idea 1: Look at the denominator – or amount of cancelation to get to this denominator. Leads to the study of

$$F(x, y) := \#\{n \leq x : \gcd(n, \sigma(n)) > y\}.$$

One can show, e.g., that for fixed $\alpha \in (0, 1)$,

$$F(x, x^\alpha) = x^{1-\alpha+o(1)}$$

as $x \to \infty$.

Idea 2: Use continued fractions. Let $L(\alpha)$ denote the length of the continued fraction expansion of the rational number α. How large is $L(\sigma(N)/N)$?

Leads one to study

$$G(x, k) = \#\{n \leq x : L(\sigma(n)/n) = k\}.$$

Clearly $G(x, 1)$ counts "multiply perfect numbers".

Theorem

We have $G(x, 1) \leq x^{W'/\log\log x}$ *for* $x > 3$.

Clearly $\sigma(p)/p = 1 + 1/p$ has length 2.

Theorem

We have $G(x, 2) \sim \pi(x)$ as $x \to \infty$.

What is the situation with $G(x, 3)$? Note that if $N = 28p$ with $p > 7$, then

$$\sigma(N)/N = 2 + \frac{1}{(p-1)/2 + \frac{1}{2}}$$

has length 3. Much we (I) don't know:

- Is $G(x, 3) \ll x/\log x$.
- Is each $G(x, k) \ll x/\log x$ for fixed k? What about $\gg x/\log x$?
- What is the average order of the arithmetic function $L(\sigma(N)/N)$? the normal order?

Thanks for your attention!