Sequences and Series:

An Introduction to Mathematical Analysis

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Contents

Chapter 1

Sequences

1.1 The general concept of a sequence

We begin by discussing the concept of a sequence. Intuitively, a sequence is an ordered list of objects or events. For instance, the sequence of events at a crime scene is important for understanding the nature of the crime. In this course we will be interested in sequences of a more mathematical nature; mostly we will be interested in sequences of numbers, but occasionally we will find it interesting to consider sequences of points in a plane or in space, or even sequences of sets.

Let's look at some examples of sequences.

Example 1.1.1

Emily flips a quarter five times, the sequence of coin tosses is HTTHT where H stands for "heads" and T stands for "tails".

As a side remark, we might notice that there are $2^5 = 32$ different possible sequences of five coin tosses. Of these, 10 have two heads and three tails. Thus the probability that in a sequence of five coin tosses, two of them are heads and three are tails is 10/32, or 5/16. Many probabilistic questions involve studying sets of sequences such as these.

Example 1.1.2

John picks colored marbles from a bag, first he picks a red marble, then a blue one, another blue one, a yellow one, a red one and finally a blue one. The sequence of marbles he has chosen could be represented by the symbols RBBYRB.

Example 1.1.3

Harry the Hare set out to walk to the neighborhood grocery. In the first ten minutes he walked half way to the grocery. In the next ten minutes he walked half of the remaining distance, so now he was $3/4$ of the way to the grocery. In the following ten minutes he walked half of the remaining distance again, so now he has managed to get $7/8$ of the way to the grocery. This sequence of events continues for some time, so that his progress follows the pattern $1/2$, $3/4$, $7/8$, $15/16$, $31/32$, and so on. After an hour he is $63/64$ of the way to the grocery. After two hours he is $4095/4096$ of the way to the grocery. If he was originally one mile from the grocery, he is now about 13 inches away from the grocery. If he keeps on at this rate will he ever get there? This brings up some pretty silly questions; For instance, if Harry is 1 inch from the grocery has he reached it yet? Of course if anybody manages to get within one inch of their goal we would usually say that they have reached it. On the other hand, in a race, if Harry is 1 inch behind Terry the Tortoise he has lost the race. In fact, at Harry's rate of deceleration, it seems that it will take him forever to cross the finish line.

Example 1.1.4

Harry's friend Terry the Tortoise is more consistent than Harry. He starts out at a slower pace than Harry and covers the first half of the mile in twenty minutes. But he covers the next quarter of a mile in 10 minutes and the next eighth of a mile in 5 minutes. By the time he reaches $63/64$ of the mile it has taken less than 40 minutes while it took Harry one hour. Will the tortoise beat the hare to the finish line? Will either of them ever reach the finish line? Where is Terry one hour after the race begins?

Example 1.1.5

Build a sequence of numbers in the following fashion. Let the first two numbers of the sequence be 1 and let the third number be $1 + 1 = 2$. The fourth number in the sequence will be $1 + 2 = 3$ and the fifth number is $2 + 3 = 5$. To continue the sequence, we look for the previous two terms and add them together. So the first ten terms of the sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

This sequence continues forever. It is called the *Fibonnaci* sequence. This sequence is said to appear ubiquitously in nature. The volume of the chambers of the nautilus shell, the number of seeds in consecutive rows of a sunflower, and many natural ratios in art and architecture are purported to progress

by this natural sequence. In many cases the natural or biological reasons for this progression are not at all straightforward.

The sequence of natural numbers,

$$1, 2, 3, 4, 5, ...$$

and the sequence of odd natural numbers,

$$1, 3, 5, 7, 9, ...$$

are other simple examples of sequences that continue forever. The symbol ... (called ellipses) represents this infinite continuation. Such a sequence is called an *infinite sequence*. In this book most of our sequences will be infinite and so from now on when we speak of sequences we will mean infinite sequences. If we want to discuss some particular *finite sequence* we will specify that it is finite.

Since we will want to discuss general sequences in this course it is necessary to develop some notation to represent sequences without writing down each term explicitly. The fairly concrete notation for representing a general infinite sequence is the following:

$$a_1, a_2, a_3, ...$$

where a_1 represents the first number in the sequence, a_2 the second number, and a_3 the third number, etc. If we wish to discuss an entry in this sequence without specifying exactly which entry, we write a_i or a_j or some similar term.

To represent a finite sequence that ends at, say, the 29th entry we would write

$$a_1, a_2, ..., a_{29}.$$

Here the ellipses indicate that there are several intermediate entries in the sequence which we don't care to write out explicitly. We may also at times need to represent a series that is finite but of some undetermined length; in this case we will write

$$a_1, a_2, ..., a_N$$

where N represents the fixed, but not explicitly specified length.

A slightly more sophisticated way of representing the abstract sequence $a_1, a_2, ...$ is with the notation:

$$\{a_i\}_{i=1}^{\infty}.$$

The finite sequence $a_1, a_2, ..., a_N$ is similarly represented by:

$$\{a_i\}_{i=1}^{N}.$$

Since in this text we study mostly infinite sequences, we will often abbreviate $\{a_i\}_{i=1}^{\infty}$ with simply $\{a_i\}$. Although this looks like set notation you should be careful not to confuse a sequence with the set whose elements are the entries of the sequence. A set has no particular ordering of its elements but a sequence certainly does. For instance, the sequence $1, 1, 1, 1, ...$ has infinitely many terms, yet the set made of these terms has only one element.

When specifying any particular sequence, it is necessary to give some description of each of its terms. This can be done in various ways. For a (short) finite sequence, one can simply list the terms in order. For example, the sequence $3, 1, 4, 1, 5, 9$ has six terms which are easily listed. On the other hand, these are the first six terms of the decimal expansion of π, so this sequence can be extended to an infinite sequence, $3, 1, 4, 1, 5, 9, ...$, where it is understood from the context that we continue this sequence by computing further terms in the decimal expansion of π. Here are a few other examples of infinite sequences which can be inferred by listing the first few terms:

$$1, 2, 3, 4, ...$$
$$2, 4, 6, 8, ...$$
$$5, 10, 15, 20, ...$$
$$1, 1, 2, 3, 5, 8, 13, ...$$

Well maybe it is not so obvious how to extend this last sequence unless you are familiar with the Fibonacci sequence discussed in Example 1.1.5. This last example demonstrates the drawback of determining a sequence by inference, it leaves it to the reader to discover what method you used to determine the next term.

A better method of describing a sequence is to state how to determine the nth term with an *explicit formula* . For example, the sequence $1, 2, 3, 4, ...$ is easily specified by saying $a_n = n$. Formulas for the second and third sequence above can be specified with the formulas $a_n = 2n$ and $a_n = 5n$ respectively. An explicit formula for the nth term of the Fibonacci sequence, or the nth term in the decimal expansion of π is not so easy to find. In exercise 1.2.21 we will find an explicit formula for the Fibonacci sequence, but there is no such explicit formula for the nth term in the decimal expansion of π.

Example 1.1.6

The nth term in a sequence is given by $a_n = (n^2 + n)/2$. The first five terms are $1, 3, 6, 10, 15$.

Example 1.1.7

The nth term in the sequence $\{b_n\}$ is given by $b_n = 1 - \frac{1}{n^2}$. The first six terms of this sequence are

$$0, 3/4, 8/9, 15/16, 24/25, 35/36.$$

A third way of describing a sequence is through a *recursive formula*. A recursive formula describes the nth term of the sequence in terms of previous terms in the sequence. The easiest form of a recursive formula is a description of a_n in terms of a_{n-1}. Many of our earlier examples of numerical sequences were described in this way.

Example 1.1.8

Let's return to Example 1.1.3 above. Each 10 minutes, Harry walks half of the remaining distance to the neighborhood. Let's denote the fraction of the total distance that Harry has travelled after n chunks of ten minutes by a_n. So $a_1 = 1/2, a_2 = 3/4, a_3 = 7/8$, etc. Then the fraction of the total distance that remains to be travelled after n chunks of ten minutes is $1 - a_n$. Since the distance travelled in the next ten minutes is half of this remaining distance, we see that

$$a_{n+1} = a_n + \frac{1}{2}(1 - a_n) = \frac{1}{2}(1 + a_n).$$

Notice that this formula is not enough by itself to determine the sequence $\{a_n\}$. We must also say how to start the sequence by supplying the information that

$$a_1 = 1/2.$$

Now, with this additional information, we can use the formula to determine further terms in the sequence:

$$a_2 = \frac{1}{2}(1 + a_1) = \frac{1}{2}(1 + 1/2) = 3/4$$
$$a_3 = \frac{1}{2}(1 + a_2) = \frac{1}{2}(1 + 3/4) = 7/8$$
$$a_4 = \frac{1}{2}(1 + a_3) = \frac{1}{2}(1 + 7/8) = 15/16,$$

etc.

Example 1.1.9

Let's have another look at the Fibonacci sequence from Example 1.1.5 above. Here the nth term is determined by two previous terms, indeed

$$a_{n+1} = a_n + a_{n-1}.$$

Now we can't get started unless we know the first *two* steps in the sequence, namely a_1 and a_2. Since we are told that $a_1 = 1$ and $a_2 = 1$ also, we can use the recursion formula to determine

$$a_3 = a_2 + a_1 = 1 + 1 = 2.$$

And now since we have both a_2 and a_3 we can determine

$$a_4 = a_3 + a_2 = 2 + 1 = 3,$$

and similarly

$$a_5 = a_4 + a_3 = 3 + 2 = 5,$$
$$a_6 = a_5 + a_4 = 5 + 3 = 8,$$
$$a_7 = a_6 + a_5 = 8 + 5 = 13,$$

and so on.

To conclude this section we mention two more families of examples of sequences which often arise in mathematics, the *arithmetic* (the accent is on the third syllable!) sequences and the *geometric* sequences.

An *arithmetic* sequence has the form $a, a+b, a+2b, a+3b, ...$ where a and b are some fixed numbers. An explicit formula for this arithmetic sequence is given by $a_n = a + (n-1)b$, $n \in \mathbb{N}$, a recursive formula is given by $a_1 = a$ and $a_n = a_{n-1} + b$ for $n > 1$. Here are some examples of arithmetic sequences, see if you can determine a and b in each case:

$$1, 2, 3, 4, 5, ...$$
$$2, 4, 6, 8, 10, ...$$
$$1, 4, 7, 10, 13, ...$$

The distinguishing feature of an arithmetic sequence is that each term is the *arithmetic mean* of its neighbors, i.e. $a_n = (a_{n-1} + a_{n+1})/2$, (see exercise 13).

A *geometric* sequence has the form $a, ar, ar^2, ar^3, ...$ for some fixed numbers a and r. An explicit formula for this geometric sequence is given by $a_n = ar^{n-1}, n \in \mathbb{N}$. A recursive formula is given by $a_1 = a$ and $a_n = ra_{n-1}$ for $n > 1$. Here are some examples of geometric sequences, see if you can determine a and r in each case:

$$2, 2, 2, 2, 2...$$
$$2, 4, 8, 16, 32, ...$$
$$3, 3/2, 3/4, 3/8, 3/16, ...$$
$$3, 1, 1/3, 1/9, 1/27, ...$$

Geometric sequences (with positive terms) are distinguished by the fact that the nth term is the *geometric mean* of its neighbors, i.e. $a_n = \sqrt{a_{n+1}a_{n-1}}$, (see exercise 14).

Example 1.1.10
If a batch of homebrew beer is inoculated with yeast it can be observed that the yeast population grows for the first several hours at a rate which is proportional to the population at any given time. Thus, if we let p_n denote the yeast population measured after n hours have passed from the inoculation, we see that there is some number $\alpha > 1$ so that

$$p_{n+1} = \alpha p_n.$$

That is, p_n forms a geometric sequence.

Actually, after a couple of days, the growth of the yeast population slows dramatically so that the population tends to a steady state. A better model for the dynamics of the population that reflects this behavior is

$$p_{n+1} = \alpha p_n - \beta p_n^2,$$

where α and β are constants determined by the characteristics of the yeast. This equation is known as the *discrete logistic equation* . Depending on the values of α and β it can display surprisingly varied behavior for the population sequence, p_n. As an example, if we choose $\alpha = 1.2, \beta = .02$ and

$p_0 = 5$, we get

$$p_1 = 5.5$$
$$p_2 = 5.995$$
$$p_3 = 6.475199500$$
$$p_4 = 6.931675229$$
$$p_5 = 7.357047845$$
$$p_6 = 7.745934354$$
$$p_7 = 8.095131245$$
$$p_8 = 8.403534497$$
$$p_9 = 8.671853559$$
$$p_{10} = 8.902203387.$$

Further down the road we get $p_{20} = 9.865991756, p_{30} = 9.985393020, p_{40} = 9.998743171$, and $p_{100} = 9.999999993$. Apparently the population is leveling out at about 10. It is interesting to study the behavior of the sequence of p_n's for other values of α and β (see exercise 15).

EXERCISES 1.1

1. a.) How many sequences of six coin tosses consist of three heads and three tails?

    b.) How many different sequences of six coin tosses are there altogether?

    c.) In a sequence of six coin tosses, what is the probability that the result will consist of three heads and three tails?

2. (For students with some knowledge of combinatorics.) In a sequence of $2n$ coin tosses, what is the probability that the result will be exactly n heads and n tails?

3. Let $\{a_n\}$ be the sequence given explicitly by $a_n = 2n - 1$. Write out a_1, a_2, a_3, a_4, and a_5. Describe this sequence in words.

4. a.) Let the sequence $\{a_n\}$ be given by the recursive relation $a_{n+1} = \frac{3+a_n}{2}$ with $a_1 = 1$. Write down the first five terms of the sequence.

   b.) Let the sequence $\{a_n\}$ be given by the recursive relation $a_{n+1} = \frac{3+a_n}{2}$ with $a_1 = 0$. Write down the first five terms of the sequence.

5. a.) Let $\{a_n\}$ be the sequence given explicitly by $a_n = \frac{1}{2}(n^2 - n)$. Find explicitly a_1, a_2, a_5 and a_{10}.

   b.) Show that $a_{n+1} = \frac{1}{2}(n^2 + n)$, and find a formula for $a_{n+1} - a_n$.

   c.) Conclude that $a_{n+1} = a_n + n$, with $a_1 = 0$ gives a recursive formula for a_{n+1} in terms of a_n.

6. Let $\{a_n\}$ be the sequence given explicitly by $a_n = n^2$. Use the method developed in exercise 5 to find a recursive formula for a_{n+1} in terms of a_n.

7. Let $\{a_n\}$ be the sequence given explicitly by $a_n = \frac{n(n+1)(2n+1)}{6}$. Use the method developed in exercise 5 to find a recursive formula for a_{n+1} in terms of a_n.

8. a.) Let $\{a_n\}$ be the sequence of natural numbers, so $a_n = n$. Define a new sequence $\{b_n\}$ by $b_n = a_{2n-1}$. Write down explicitly b_1, b_2, b_3, and b_{10}. Give an explicit formula for b_n.

   b.) Let $\{a_n\}$ be the sequence of natural numbers, so $a_n = n$. Define a new sequence $\{c_n\}$ by $c_n = a_{n^2}$. Write down explicitly c_1, c_2, c_3, and c_{10}. Give an explicit formula for c_n.

9. Let $\{a_n\}$ be the sequence given explicitly by $a_n = n^2$. Define a new sequence $\{b_n\}$ by $b_n = a_{2n}$. Write down explicitly b_1, b_2, b_3, and b_{10}. Give an explicit formula for b_n.

10. Let $\{a_n\}$ be the sequence given recursively by

$$a_{n+1} = \frac{a_n^2 + 2}{2a_n},$$

with $a_1 = 1$. Write out the first 5 terms of this sequence. First find them as fractions and then, using a calculator, give 5-place decimal approximations to them. Compare these numbers to the decimal expansion for $\sqrt{2}$.

11. (Thanks to Mo Hendon.) Joe is trying to sell his old car for $1000 and Mo offers him $500. Joe says, "Ok, let's split the difference, I'll sell it to you for $750." But Mo says, "That is still too much, but I'll offer to split the difference now and pay you $625." Joe and Mo continue to dicker in this manner for a long time. Write down a recursive formula for the nth offer in terms of the previous two offers. Do you think they can ever settle on a price?

12. Little Johnny is at home and decides he wants to go hang out at the local pool hall. So he sets off toward the pool hall, but when he is halfway there, he remembers that his mother told him not to go there anymore and turns back towards home. When he is halfway from the midpoint back to his home, he changes his mind once more and heads to the pool hall. Continuing in this way, Johnny changes his mind whenever he has traversed half of the distance from his turning point to the new destination. If D is the distance from Johnny's house to the pool hall, what fractions of D are the distance from his house to the first four turning points? What happens to Johnny eventually?

13. a.) Let an arithmetic sequence be given by the recursive formula $a_1 = a$ and $a_n = a_{n-1} + b$, $n > 1$ and prove that $a_n = (a_{n+1} + a_{n-1})/2$ for all $n > 1$.

    b.) Suppose that the arithmetic sequence is given by the explicit formula. $a_n = a + (n-1)b$. Again, show that $a_n = (a_{n+1} + a_{n-1})/2$ for all $n > 1$.

> **Comment on Proving Equalities:**
> When proving two quantities are equal, as in the above exercise, it is usually best to begin with the expression on one side of the equality and manipulate that expression with algebra until you arrive at the other side of the original equality. Logically, it doesn't matter which side you start on, as long as you progress directly to the other side. The choice is yours, but often you will find that one direction seems to be easier to follow than the other. In fact, usually when you are figuring out how to prove an equality you will start off playing with both sides until you see what is going on, but in the end, the proof should not be written up

in that fashion. For example, in part a of the above exercise, you may wish to start with the expression $(a_{n+1} + a_{n-1})/2$ and substitute for the term a_{n+1} the fact that $a_{n+1} = a_n + b$. Then use the fact that $b + a_{n-1} = a_n$ to see that the original expression is equal to $(a_n + a_n)/2$. To write this up formally, one might say:

Replacing n with $n + 1$ in the expression $a_n = a_{n-1} + b$ we see that $a_{n+1} = a_n + b$. Thus we see that $(a_{n+1} + a_{n-1})/2 = (a_n + b + a_{n-1})/2$. Furthermore, since $a_{n-1} + b = a_n$, we can conclude that $(a_n + b + a_{n-1})/2 = (a_n + a_n)/2 = a_n$. Combining the first and last equalities yields

$$(a_{n+1} + a_{n-1})/2 = (a_n + b + a_{n-1})/2 = (a_n + a_n)/2 = a_n.$$

14. a.) Let a and r be positive real numbers and define a geometric sequence by the recursive formula $a_1 = a$ and $a_n = ra_{n-1}, n > 1$. Show that $a_n = \sqrt{a_{n+1}a_{n-1}}$ for all $n > 1$.

   b.) Again with a and r positive real numbers define a geometric sequence by the explicit formula $a_n = ar^{n-1}$. Here also, show that $a_n = \sqrt{a_{n+1}a_{n-1}}$ for all $n > 1$.

15. (Calculator needed.) Find the first 20 terms of the sequences given by $p_{n+1} = \alpha p_n - \beta p_n^2$ where α, β, and p_0 are given below. In each case write a sentence or two describing what you think the long term behavior of the population will be.

   a.) $\alpha = 2$, $\beta = 0.1$, $p_0 = 5$.

   b.) $\alpha = 2.8$, $\beta = 0.18$, $p_0 = 5$.

   c.) $\alpha = 3.2$, $\beta = 0.22$, $p_0 = 5$.

   d.) $\alpha = 3.8$, $\beta = 0.28$, $p_0 = 5$.

   If you have MAPLE available you can explore this exercise by changing the values of a and b in the following program:

```
[> restart:
[> a:=1.2; b:= 0.02;
[> f:=x->a*x - b*x^2;
[> p[0]:=5;
[> for j from 1 to 20 do p[j]:=f(p[j-1]); end do;
```

1.2   The sequence of natural numbers

A very familiar and fundamental sequence is that of the *natural numbers,* $\mathbb{N} = \{1, 2, 3, ...\}$. As a sequence, we might describe the natural numbers by the explicit formula

$$a_n = n$$

but this seems circular. (It certainly does not give a definition of the natural numbers.) Somewhat more to the point is the recursive description,

$$a_{n+1} = a_n + 1, \quad a_1 = 1$$

but again this is not a definition of the natural numbers since the use of the sequence notation implicitly refers to the natural numbers already. In fact, the nature of the existence of the natural numbers is a fairly tricky issue in the foundations of mathematics which we won't delve into here, but we do want to discuss a defining property of the natural numbers that is extremely useful in the study of sequences and series:

The Principle of Mathematical Induction
Let S be a subset of the natural numbers, $\mathbb{N}$, which satisfies the following two properties:
1.) $1 \in S$
2.) if $k \in S$, then $k + 1 \in S$.
Then S must be the entire set of natural numbers, i.e. $S = \mathbb{N}$.

Example 1.2.1
    a.) Let $S = \{1, 2, 3, 4, 5\}$. Then S satisfies property 1.) but not property 2.) since 5 is an element of S but 6 is not an element of S.
    b.) Let $S = \{2, 3, 4, 5, ...\}$, the set of natural numbers greater than 1. This S satisfies property 2.) but not property 1.).
    c.) Let $S = \{1, 3, 5, 7, ...\}$, the set of odd natural numbers. Then S satisfies property 1.) but not property 2.).

    The Principle of Mathematical Induction (which we shall henceforth abbreviate by PMI) is not only an important defining property of the natural numbers, it also gives a beautiful method of proving an infinite sequence of statements. In the present context, we will see that we can use PMI to verify explicit formulae for sequences which are given recursively.

Example 1.2.2

Consider the sequence given recursively by

$$a_{n+1} = a_n + (2n+1), \quad a_1 = 1.$$

The nth term, a_n, is the sum of the first n odd natural numbers. Writing out the first 5 terms we see

$$a_1 = 1,$$
$$a_2 = a_1 + 3 = 4,$$
$$a_3 = a_2 + 5 = 9,$$
$$a_4 = a_3 + 7 = 16,$$
$$a_5 = a_4 + 9 = 25.$$

Noticing a pattern here we might conjecture that, in general, $a_n = n^2$. Here is how we can use PMI to prove this conjecture:

Let S be the subset of natural numbers for which it is true that $a_n = n^2$, i.e.

$$S = \{n \in \mathbb{N} \mid a_n = n^2\}.$$

We know that $a_1 = 1$, which is equal to 1^2, so $1 \in S$. Thus S satisfies the first requirement in PMI. Now, let k be some arbitrary element of S. Then, by the description of S we know that k is some particular natural number such that $a_k = k^2$. According to the definition of the sequence,

$$a_{k+1} = a_k + (2k+1),$$

so, since $a_k = k^2$, we can conclude that

$$a_{k+1} = k^2 + (2k+1).$$

However, since $k^2 + (2k+1) = (k+1)^2$, we conclude that

$$a_{k+1} = (k+1)^2,$$

i.e. $k+1$ *is an element of S as well.* We have just shown that if $k \in S$ then $k+1 \in S$, i.e. S satisfies the second requirement of PMI. Therefore we can conclude that $S = \mathbb{N}$, i.e. the explicit formula $a_n = n^2$ is true for every natural number n.

Remark: It is common to get confused at the last step and say $a_{k+1} \in S$ instead of $k + 1 \in S$. Remember: S is the set of *subscripts* for which the statement is true.

Comments on the Subscript k:
    In general, letters like i, j, k, l, m, and n are used to denote arbitrary or unspecified natural numbers. In different contexts these letters can represent either an *arbitrary* natural number or a *fixed but unspecified* natural number. These two concepts may seem almost the same, but their difference is an important issue when studying PMI. To illustrate this here we are using two different letters to represent the different concepts. Where the subscript n is used we are talking about an arbitrary natural number, i.e. we are claiming that for this sequence, $a_n = n^2$ for any natural number you choose to select. On the other hand, the subscript k is used above to denote a fixed but unspecified natural number, so we assume $a_k = k^2$ for that particular k and use what we know about the sequence to prove that the general formula holds for the *next* natural number, i.e. $a_{k+1} = (k+1)^2$.

Example 1.2.3
    Let $\{a_n\}$ be the sequence defined recursively by $a_{n+1} = a_n + (n+1)$, and $a_1 = 1$. Thus a_n is the sum of the first n natural numbers. Computing the first few terms, we see $a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 10$, and $a_5 = 15$. Let's now use PMI to prove that in general, $a_n = \frac{n(n+1)}{2}$. (You might want to check that this formula works in the above five cases.)

    Let S be the set of natural numbers n so that it is true that $a_n = \frac{n(n+1)}{2}$, i.e.

$$S = \{n \in \mathbb{N} \mid a_n = \frac{n(n+1)}{2}\}.$$

We wish to show that $S = \mathbb{N}$ by means of PMI. First notice that $1 \in S$ since $a_1 = 1$ and $\frac{1(1+1)}{2} = 1$ also. That is, the formula is true when $n = 1$. Now assume that we know that some particular natural number k is an element of S. Then, by the definition of S, we know that $a_k = \frac{k(k+1)}{2}$. From the recursive definition of the sequence, we know that $a_{k+1} = a_k + (k+1)$. Substituting

in $a_k = \frac{k(k+1)}{2}$, we get

$$
\begin{aligned}
a_{k+1} &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)}{2} + \frac{2k+2}{2} \\
&= \frac{k^2 + 3k + 2}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}.
\end{aligned}
$$

But this is the explicit formula we are trying to verify in the case that $n = k+1$. Thus we have proven that whenever k is a member of S then so is $k+1$ a member of S. Therefore S satisfies the two requirements in PMI and we conclude that $S = \mathbb{N}$.

Comments on Set Notation:

Although the issue of defining the notion of a *set* is a fairly tricky subject, in this text we will be concerned mostly with describing subsets of some given set (such as the natural numbers $\mathbb{N}$, or the real numbers $\mathbb{R}$) which we will accept as being given. Generally such subsets are described by a condition, or a collection of conditions. For example, the even natural numbers, let's call them E, are described as the subset of natural numbers which are divisible by 2. The notation used to describe this subset is as follows:

$$E = \{n \in \mathbb{N} \mid 2 \text{ divides } n\}.$$

There are four separate components to this notation. The brackets $\{,\}$ contain the set, the first entry (in this example $n \in \mathbb{N}$) describes the set from which we are taking a subset, the vertical line $\mid$ separates the first entry from the conditions (and is often read as "such that"), and the final entry gives the conditions that describe the subset (if there is more than one condition then they are simply all listed, with commas separating them). Of course sometimes a set can be described by two different sets of conditions, for instance an even natural number can also be described as twice another natural number. Hence we have

$$E = \{n \in \mathbb{N} \mid n = 2k \text{ for some } k \in \mathbb{N}\}.$$

There are times when one has to be a bit tricky in re-labeling indices to apply PMI exactly as stated. Here is an example of this.

Example 1.2.4

Here we use induction to prove[1] that $n^2 + 5 < n^3$ for all $n \geq 3$. If we start off as usual by letting

$$S = \{n \in \mathbb{N} \mid n^2 + 5 < n^3\}$$

we will be in big trouble since it is easy enough to see that 1 is not an element of S. So it can't possibly be true that $S = \mathbb{N}$. A formal way of getting around this problem is to shift the index in the inequality so that the statement to prove begins at 1 instead of 3. To do this, let $n = m + 2$, so that $m = 1$ corresponds to $n = 3$ and then substitute this into the expression we want to prove. That is, if we want to prove that $n^2 + 5 < n^3$ for all $n \geq 3$, it is equivalent to prove that $(m+2)^2 + 5 < (m+2)^3$ for all $m \in \mathbb{N}$. Thus we can now proceed with PMI by setting

$$\tilde{S} = \{m \in \mathbb{N} \mid (m+2)^2 + 5 < (m+2)^3\},$$

and showing that properties 1.) and 2.) hold for $\tilde{S}$. Actually, this formality is generally a pain in the neck. Instead of being so pedantic, we usually proceed as follows with the original set S:

1'.) Show that $3 \in S$.

2'.) Show that if $k \in S$ then so is $k + 1 \in S$.

Then we can conclude that $S = \{n \in \mathbb{N} \mid n \geq 3\}$. Of course, to do this we are actually applying some variation of PMI but we won't bother to state all such variations formally.

To end this example, let's see that 1'.) and 2'.) are actually true in this case. First, it is easy to see that $3 \in S$ since $3^2 + 5 = 14$ while $3^3 = 27$. Next, let us assume that some particular k is in S. Then we know that $k^2 + 5 < k^3$. But

$$\begin{aligned}
(k+1)^2 + 5 &= (k^2 + 2k + 1) + 5 \\
&= (k^2 + 5) + (2k + 1) \\
&< k^3 + (2k + 1) \\
&< k^3 + (2k + 1) + (3k^2 + k) \\
&= k^3 + 3k^2 + 3k + 1 \\
&= (k+1)^3,
\end{aligned}$$

---

[1] See the end of section 1.3 for a review of the properties of inequalities.

where the first inequality follows from the assumption that $k^2 + 5 < k^3$, and the second inequality follows from the fact that $3k^2 + k > 0$ for $k \in \mathbb{N}$.

For the next example we will need a more substantial variation on the Principle of Mathematical Induction called the Principle of Complete Mathematical Induction (which we will abbreviate PCMI).

The Principle of Complete Mathematical Induction
Let S be a subset of the natural numbers, $\mathbb{N}$, satisfying
1.) $1 \in S$
2.) if $1, 2, 3, ..., k$ are all elements of S, then $k+1$ is an element of S as well.
Then $S = \mathbb{N}$.

Example 1.2.5
Let $\{a_n\}$ be the sequence given by the two-term recursion formula $a_{n+1} = 2a_n - a_{n-1} + 2$ for $n > 1$ and $a_1 = 3$ and $a_2 = 6$. Listing the first seven terms of this sequence, we get $3, 6, 11, 18, 27, 38, 51, ...$ Perhaps a pattern is becoming evident at this point. It seems that the nth term in the sequence is given by the explicit formula $a_n = n^2 + 2$. Let's use PCMI to prove that this is the case. Let S be the subset of natural numbers n such that it is true that $a_n = n^2 + 2$, i.e.

$$S = \{n \in \mathbb{N} | \ a_n = n^2 + 2\}.$$

We know that $1 \in S$ since we are given $a_1 = 3$ and it is easy to check that $3 = 1^2 + 2$. Now assume that we know $1, 2, 3, ..., k$ are all elements of S. Now, as long as $k > 1$ we know that $a_{k+1} = 2a_k - a_{k-1} + 2$. Since we are assuming that k and $k - 1$ are in S, we can write $a_k = k^2 + 2$ and $a_{k-1} = (k-1)^2 + 2$. Substituting these into the expression for a_{k+1} we get

$$\begin{aligned} a_{k+1} &= 2(k^2 + 2) - [(k-1)^2 + 2] + 2 \\ &= 2k^2 + 4 - (k^2 - 2k + 3) + 2 \\ &= k^2 + 2k + 3 \\ &= (k+1)^2 + 2. \end{aligned} \tag{1.1}$$

Thus $k + 1 \in S$ as well. We aren't quite done yet! The last argument only works when $k > 1$. What about the case that $k = 1$? If we know that $1 \in S$ can we conclude that $2 \in S$? Well, not directly, but we can check that $2 \in S$ anyway. After all, we are given that $a_2 = 6$ and it is easy to check that

$6 = 2^2 + 2$. Notice that if we had given a different value for a_2, then the entire sequence changes from there on out so the explicit formula would no longer be correct. However, if you aren't careful about the subtlety at $k = 1$, you might think that you could prove the formula by induction no matter what is the value of a_2!

Remark:

Although hypotheses 2.) of PMI and PCMI are quite different, it turns out that these two principles are logically equivalent: Assuming that the natural numbers satisfy PMI one can prove that they also must satisfy PCMI, and vice versa, if we assume PCMI we can prove PMI. We'll outline a proof of this equivalence in exercise 24.

EXERCISES 1.2

1. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 1$, and $a_{n+1} = a_n + (n+1)^2$. (So a_n is the sum of the first n squares.) Use PMI to prove that

$$a_n = \frac{n(n+1)(2n+1)}{6}.$$

2. a.) Let $\{a_n\}$ be the sequence given recursively by $a_1 = 1$, and $a_{n+1} = a_n + (n+1)^3$. (So a_n is the sum of the first n cubes.) Use PMI to prove that

$$a_n = \left(\frac{n(n+1)}{2}\right)^2.$$

   b.) Combine the result from part a.) with the result in Example 1.2.3 to prove the remarkable fact that

$$(1 + 2 + 3 + \ldots + n)^2 = 1^3 + 2^3 + 3^3 + \ldots + n^3.$$

3. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 2$, and $a_{n+1} = a_n + 2(n+1)$. Find a formula for a_n and use induction to prove that it is true for all n.

4. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 5$, and $a_{n+1} = a_n + 2n + 3$. Find a formula for a_n and use induction to prove that it is true for all n.

5. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 1$, and $a_{n+1} = a_n + 2^n$. (So a_n is the sum of the first n powers of 2.) Use induction to show that

$$a_n = 2^n - 1.$$

In mathematics the word "show" is used synonymously with "prove".

6. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 1$, and $a_{n+1} = a_n + 3^n$. (So a_n is the sum of the first n powers of 3.) Use induction to show that

$$a_n = \frac{3^n - 1}{2}.$$

7. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 1$, and $a_{n+1} = a_n + r^n$, where r is a fixed real number which is not equal to 1. (So a_n is the sum of the first n powers of r.) Use induction to show that

$$a_n = \frac{r^n - 1}{r - 1}.$$

8. Let c, r, and a_0 be fixed real numbers and define the sequence $\{a_n\}$ recursively by $a_{n+1} = c + ra_n$. Show by induction that

$$a_n = \left(\frac{r^n - 1}{r - 1}\right) c + r^n a_0.$$

9. Let r be a real number with $0 < r < 1$. Prove by induction that $0 < r^n < 1$ for every $n \in \mathbb{N}$. (You may want to use the fact that if $0 < a < 1$ and $0 < b < 1$, then $0 < ab < 1$. See section 1.3.)

10. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 1$, and $a_{n+1} = a_n - \frac{1}{n(n+1)}$. Show by induction that

$$a_n = \frac{1}{n}.$$

11. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 1/2$, and $a_{n+1} = a_n + \frac{1}{(n+1)(n+2)}$. Show by induction that

$$a_n = \frac{n}{n + 1}.$$

12. Prove by induction that $2n + 1 \leq 3n^2$ for all $n \in \mathbb{N}$.

13. Prove by induction that $2n^2 - 1 \leq n^3$ for all $n \in \mathbb{N}$. (Hint: You may need the result in problem 12.)

14. a.) Prove by induction that $2^n < n!$ for all $n \geq 4$.

    b.) Use induction to prove that $\dfrac{2^n}{n!} < \dfrac{1}{n}$ for $n > 5$.

15. Let $\{a_n\}$ be the sequence defined recursively by $a_1 = 2$, and $a_{n+1} = \dfrac{2a_n + 1}{a_n + 2}$, for $n \geq 1$. Prove by induction that $a_n \geq 1$ for all $n \in \mathbb{N}$. (*Hint:* Look at $a_{n+1} - 1$.)

16. Let $\{a_n\}$ be the sequence defined recursively by $a_1 = 3$, and $a_{n+1} = \dfrac{3a_n + 4}{a_n + 3}$, for $n \geq 1$. Prove by induction that $a_n \geq 2$ for all $n \in \mathbb{N}$.

17. Let S be a set with n elements. Prove by induction that S has 2^n subsets.

18. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 1, a_2 = 8$, and $a_{n+1} = 2a_n - a_{n-1} + 6n$ for $n > 1$. Using complete induction show that

$$a_n = n^3.$$

19. Let $\{a_n\}$ be the sequence given recursively by $a_1 = 1, a_2 = 4, a_3 = 9$, and $a_{n+1} = 3a_n - 3a_{n-1} + a_{n-2}$ for $n > 2$. Using complete induction show that

$$a_n = n^2.$$

20. What is wrong with the following argument?

Old MacDonald claims that all cows have the same color. First of all, if you have just one cow, it certainly has the same color as itself. Now, using PMI, assume that all the cows in any collection of k cows have the same color and look at a collection of $k + 1$ cows. Removing one cow from that collection leaves a collection of k cows, which must therefore all have the same color. Putting back the removed cow, and removing a different cow leaves another collection of k cows which all have the same color. Certainly then, the original collection of $k + 1$ cows must all have

the same color. By PMI, all of the cows in any finite collection of cows have the same color.

21. Let $\{a_n\}$ be the Fibonacci sequence given recursively by $a_1 = 1, a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n > 1$ (see Example 1.1.5). Using complete induction show that a_n is given explicitly by

$$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right).$$

22. (On arithmetic sequences.)

a.) Suppose that a sequence is given recursively by $a_1 = a$ and $a_n = a_{n-1} + b$ for $n > 1$, where a and b are fixed real numbers. Prove that $a_n = a + (n-1)b$ for all $n \in \mathbb{N}$.

b.) Suppose that the sequence $\{a_n\}$ satisfies $a_n = (a_{n+1} + a_{n-1})/2$ for $n > 1$. Prove that there is a $b \in \mathbb{R}$ so that $a_n = a_{n-1} + b$ for all $n > 1$. (Hint: Use the expression $a_n = (a_{n+1} + a_{n-1})/2$ to show that $a_{n+1} - a_n = a_n - a_{n-1}$ for $n > 1$ then use induction to show that $a_n - a_{n-1} = a_2 - a_1$ for all $n > 1$.)

23. (On geometric sequences.)

a.) Suppose that a sequence is given recursively by $a_1 = a$ and $a_n = r a_{n-1}$ for $n > 1$. Prove that $a_n = ar^{n-1}$ for all $n \in \mathbb{N}$.

b.) Suppose that the sequence $\{a_n\}$ satisfies $a_n = \sqrt{a_{n+1} a_{n-1}}$ for all $n > 1$. Prove that there is an $r \in \mathbb{R}$ so that $a_n = r a_{n-1}$ for all $n > 1$. (Hint: Start with the case that $a_n \neq 0$ for all n, and use the expression $a_n = \sqrt{a_{n+1} a_{n-1}}$ to show that $a_{n+1}/a_n = a_n/a_{n-1}$ for $n > 1$ then use induction to show that $a_n/a_{n-1} = a_2/a_1$ for all $n > 1$. Don't forget to separately deal with the case that some a_n is zero.)

24. In this exercise we outline the proof of the equivalence of PMI and PCMI.

a.) First we assume that the natural numbers satisfy PCMI and we take a subset $T \subset \mathbb{N}$ which satisfies hypotheses 1.) and 2.) of PMI, i.e. we know:

    1.) $1 \in T$

and that

2.) if $k \in T$ then $k + 1 \in T$.

Show directly that T also satisfies the hypotheses of PCMI, hence by our assumption we conclude that $T = \mathbb{N}$.

b.) This direction is a little trickier. Assume that the natural numbers satisfy PMI and take $T \subset \mathbb{N}$ which satisfies hypotheses 1.) and 2.) of PCMI. Now define S to be the subset of $\mathbb{N}$ given by $S = \{n \in T \mid 1, 2, ..., n \in T\}$. Prove by PMI that $S = \mathbb{N}$.

25. (Back to Fibonacci)

Here is a way to discover the formula for the Fibonacci sequence given in exercise 21.

a.) First, make the guess that the Fibonacci sequence should have geometric growth, so $a_n = \alpha r^n$ for some real numbers r and α. Use the recursive formula $a_{n+1} = a_n + a_{n-1}$ to show that

$$r = \frac{1 \pm \sqrt{5}}{2}.$$

b.) Since this gives two choices, try looking for real numbers α and β so that $a_n = \alpha \left(\dfrac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\dfrac{1 - \sqrt{5}}{2} \right)^n$. Using $a_1 = a_2 = 1$ show that

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1,$$

and

$$\alpha \left(\frac{3 + \sqrt{5}}{2} \right) + \beta \left(\frac{3 - \sqrt{5}}{2} \right) = 1.$$

c.) Now solve the above system of two equations to show $\alpha = -\beta = \dfrac{1}{\sqrt{5}}$.

(Note: This still doesn't prove that the formula in exercise 21 works for all n. You still need to do the induction argument!)

1.3 Sequences as functions

Let A and B be two arbitrary sets. A *function* from A to B, usually denoted by $f : A \to B$, is an assignment to each element a in A an element $f(a)$ in B. The set A is called the *domain* of f. Since the nth term of a sequence, a_n, is a real number which is assigned to the natural number n, a sequence can be thought of as a function from $\mathbb{N}$ to $\mathbb{R}$, and vice versa. Thus we have

Definition 1.3.1

Let $f : \mathbb{N} \to \mathbb{R}$ be a function. The *sequence defined*[2] *by* f is the sequence whose n^{th} term is given by $a_n = f(n)$. On the other hand, if a sequence $\{a_n\}$ is given, then the function $f : \mathbb{N} \to \mathbb{R}$, given by $f(n) = a_n$ is called the *function defined by* $\{a_n\}$.

Example 1.3.2

a.) The function given by $f(n) = n^2$ defines the sequence with $a_n = n^2$.

b.) The sequence given by $a_n = 1/n$ defines the function $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = 1/n$.

Remark:

If a sequence is given by an explicit formula, then that formula gives an explicit formula for the corresponding function. But just as sequences may be defined in non-explicit ways, so too can functions. For instance, the consecutive digits in the decimal expansion of π define a sequence and therefore a corresponding function, $f : \mathbb{N} \to \mathbb{R}$, yet we know of no simple explicit formula for $f(n)$.

Thinking of sequences in terms of their corresponding functions gives us an important method of visualizing sequences, namely by their *graphs*. Since the natural numbers $\mathbb{N}$ are a subset of the real numbers $\mathbb{R}$ we can graph a function $f : \mathbb{N} \to \mathbb{R}$ in the plane $\mathbb{R}^2$ but the graph will consist of only isolated points whose x-coordinates are natural numbers. In *figure* 1.3.1 we depict a graph of the first five terms of the sequence given by $a_n = 1/n$.

The interpretation of sequences as functions leads to some of the following nomenclature. (Although we state these definitions in terms of sequences, a similar definition is meant to be understood for the corresponding functions.)

---

[2]More pedantically, a sequence should be defined as such a function. However, we prefer to treat the sequence and the function as separate entities, even though we never gave the notion of a sequence a separate definition.
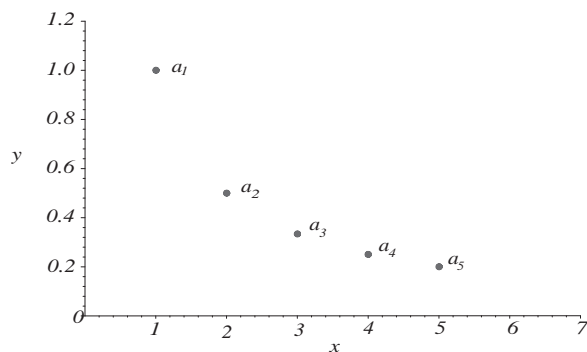
figure 1.3.1

Definition 1.3.3

The sequence $\{a_n\}$ is said to be:

i.) *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$,

ii.) *strictly increasing* if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$,

iii.) *decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$,

iv.) *strictly decreasing* if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

Remark:

1.) It is important to notice that in these definitions the condition must hold for *all* $n \in \mathbb{N}$. So if a sequence sometimes increases and sometimes decreases, then it is neither increasing nor decreasing (unless it is constant).

2.) For the rest of this text we will be making extensive use of inequalities and absolute values. For this reason we have included the basic properties of these at the end of this section. Now might be a good time to review those properties.

Example 1.3.4

a.) The sequence given by $a_n = n^2$ is strictly increasing since $(n+1)^2 = n^2 + 2n + 1 > n^2$ for all $n > 0$.

b.) The sequence given by $a_n = 0$ for all n is both increasing and decreasing but it is not strictly increasing nor strictly decreasing.

c.) A strictly increasing sequence is, of course, increasing, but it is not decreasing.

d.) The sequence given by $a_n = 3 + (-1)^n/n$ is neither increasing nor decreasing.

Example 1.3.5

Consider the sequence given recursively by

$$a_{n+1} = \frac{4a_n + 3}{a_n + 2}$$

with $a_1 = 4$. We claim that this sequence is decreasing. To check this we will show that $a_{n+1} - a_n$ is negative. Now

$$
\begin{aligned}
a_{n+1} - a_n &= \frac{4a_n + 3}{a_n + 2} - a_n \\
&= \frac{4a_n + 3}{a_n + 2} - \frac{a_n^2 + 2a_n}{a_n + 2} \\
&= \frac{2a_n + 3 - a_n^2}{a_n + 2} \\
&= \frac{-(a_n - 3)(a_n + 1)}{a_n + 2}
\end{aligned}
$$

so $a_{n+1} - a_n$ is negative as long as a_n is greater than 3. But an easy induction argument shows that $a_n > 3$ for all n since

$$
\begin{aligned}
a_{k+1} - 3 &= \frac{4a_k + 3}{a_k + 2} - 3 \\
&= \frac{4a_k + 3}{a_k + 2} - \frac{3a_k + 6}{a_k + 2} \\
&= \frac{a_k - 3}{a_k + 2}
\end{aligned}
$$

which is positive as long as $a_k > 3$.

Comment on Inequalities

In the above proof we have twice used the fact that when trying to prove that $a > b$ it is often easier to prove that $a - b > 0$.

At times we need to talk about sequences that are either decreasing or increasing without specifying which, thus we have

Definition 1.3.6

The sequence $\{a_n\}$ is said to be *monotonic* if it is either increasing or decreasing.

The following equivalent condition for an increasing sequence is often useful when studying properties of such sequences. Of course there are similar equivalent conditions for decreasing sequences and strictly increasing or decreasing sequences.

Proposition 1.3.7

A sequence, $\{a_n\}$, is increasing if and only if $a_m \geq a_n$ for all natural numbers m and n with $m \geq n$.

Proof: There are two implications to prove here. First we prove that if the sequence $\{a_n\}$ satisfies the condition that $a_m \geq a_n$ whenever $m \geq n$ then that sequence must be increasing. But this is easy, since if $a_m \geq a_n$ whenever $m \geq n$, then in particular we know that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, i.e. the sequence is increasing.

Next we must prove that if the sequence $\{a_n\}$ is known to be increasing then it must satisfy the condition $a_m \geq a_n$ whenever $m \geq n$. We will prove this by a fairly tricky induction argument. Fix a particular $n \in \mathbb{N}$ and let

$$S_n = \{l \in \mathbb{N} \mid a_{n+l} \geq a_n\}.$$

We will use PMI to show that $S_n = \mathbb{N}$. First, since the sequence is increasing we know that $a_{n+1} \geq a_n$ so $1 \in S_n$. Now suppose that some specific k is in S_n. By the definition of S_n this tells us that

$$a_{n+k} \geq a_n.$$

Since the sequence is increasing we also know that

$$a_{n+k+1} \geq a_{n+k}.$$

Putting these inequalities together yields

$$a_{n+k+1} \geq a_n,$$

i.e. $k + 1 \in S_n$. This proves, by PMI that $S_n = \mathbb{N}$. In other words, for this particular value of n we now know that $a_m \geq a_n$ whenever $m \geq n$. Since n was chosen to be an arbitrary member of $\mathbb{N}$ we conclude that $a_m \geq a_n$ for all natural numbers m and n with $m \geq n$.

$\square$

Another property of sequences that seems natural when thinking of sequences as functions is the notion of *boundedness*.

Definition 1.3.8

The sequence $\{a_n\}$ is *bounded above* if there is a real number U such that $a_n \leq U$ for all $n \in \mathbb{N}$. Any real number U which satisfies this inequality for all $n \in \mathbb{N}$ is called an *upper bound* for the sequence $\{a_n\}$.

Definition 1.3.9

The sequence $\{a_n\}$ is *bounded below* if there is a real number L such that $L \leq a_n$ for all $n \in \mathbb{N}$. Any real number L which satisfies this inequality for all $n \in \mathbb{N}$ is called a *lower bound* for the sequence $\{a_n\}$.

Example 1.3.10

It is an important property of the real numbers that *the sequence of natural numbers,* $\mathbb{N}$*, is not bounded above.* (The proof of this fact, called tthe *Archimedean Property* of the real numbers, will have to wait until exercise 1.6.13.) Thus, if r is a real number, there is always some natural number $n \in \mathbb{N}$ so that $n > r$. Of course, the natural numbers *are* bounded below by 0.

Example 1.3.11

a.) The sequence given by $a_n = n^2$, $n \in \mathbb{N}$, is bounded below by 0 but it is not bounded above.

b.) The sequence given by $a_n = -n$, $n \in \mathbb{N}$, is bounded above by 0 but it is not bounded below.

c.) The sequence given by $a_n = \frac{1}{n}$, $n \in \mathbb{N}$, is bounded below by 0 and bounded above by 1.

d.) The sequence given by $a_n = \cos(n)$, $n \in \mathbb{N}$, is bounded below by -1 and bounded above by 1.

Definition 1.3.12

The sequence $\{a_n\}$ is *bounded* if it is both bounded above and bounded below.

In the above example, only the sequences in parts c.) and d.) are bounded. In the definition of boundedness the upper and lower bounds are not necessarily related, however, it is sometimes useful to note that they can be.

Proposition 1.3.13

If the sequence $\{a_n\}$ is bounded then there is a real number $M > 0$ such that $-M \leq a_n \leq M$ for all $n \in \mathbb{N}$.

Proof: Let U be an upper bound and L a lower bound for the bounded sequence $\{a_n\}$. Let M be the maximum of the two numbers $|L|$ and $|U|$. Then since $-|L| \leq L$ and $U \leq |U|$ we have

$$-M \leq -|L| \leq L \leq a_n \leq U \leq |U| \leq M$$

for all $n \in \mathbb{N}$.

$\square$

Given two real valued functions we can add or multiply them by adding or multiplying their values. In particular, if we have $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$, we define two new functions, $(f + g)$ and fg by

$$(f + g)(n) = f(n) + g(n)$$

and

$$fg(n) = f(n)g(n).$$

If we use f and g to define sequences $\{a_n\}$ and $\{b_n\}$, i.e. $a_n = f(n)$ and $b_n = g(n)$, then the sequence given by $(f + g)$ has terms given by $a_n + b_n$ and the sequence given by fg has terms given by $a_n b_n$. These two new sequences are called the *sum* and *product* of the original sequences.

In general, one can also compose two functions as long as the set of values of the first function is contained in the domain of the second function. Namely, if $g : A \to B$ and $f : B \to C$, then we can define $f \circ g : A \to C$ by $f \circ g(a) = f(g(a))$ for each $a \in A$. Notice that $g(a) \in B$ so $f(g(a))$ makes sense. To apply the notion of composition to sequences we need to remark that the corresponding functions always have the domain given by $\mathbb{N}$. Thus, if we wish to compose these functions then the first one must take its values in $\mathbb{N}$, i.e. the first sequence must be a sequence of natural numbers. In the next few sections we will be particularly interested in the case that the first sequence is a strictly increasing sequence of natural numbers, in this case we call the composition a *subsequence* of the second sequence.

Definition 1.3.14
Let $\{a_n\}$ be the sequence defined by the function $f : \mathbb{N} \to \mathbb{R}$ and let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function (with values in $\mathbb{N}$). Then the sequence $\{b_n\}$ defined by the function $f \circ g : \mathbb{N} \to \mathbb{R}$, i.e. $b_n = f(g(n))$, is called a *subsequence* of the sequence $\{a_n\}$.

Example 1.3.15

   The sequence given by $b_n = (2n + 1)^2$ is a subsequence of the sequence given by $a_n = n^2$. If we let $f : \mathbb{N} \to \mathbb{R}$ be given by $f(n) = n^2$ and $g : \mathbb{N} \to \mathbb{N}$ be given by $g(n) = 2n + 1$, then the function corresponding to b_n is given by $f \circ g$ since $f \circ g(n) = (2n + 1)^2$.

   A good way to think about subsequences is the following. First list the elements of your original sequence in order:

$$a_1, a_2, a_3, a_4, a_5, \dots.$$

The first element, b_1, of a subsequence can by any ane of the above list, but the next element, b_2, must lie to the right of b_1 in this list. Similarly, b_3 must lie to right of b_2 in the list, b_4 must lie to the right of b_3, and so on. Of course, each of the b_j's must be taken from the original list of a_n's. Thus we can line up the b_j's under the corresponding a_n's:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \dots.$$

$$b_1 \qquad b_2 \qquad b_3$$

   A useful result which helps us review some of the above definitions is the following.

Proposition 1.3.16

   If $\{a_n\}$ is an increasing sequence, and $\{b_n\}$ is a subsequence, then $\{b_n\}$ is increasing also.

Proof: Let $\{a_n\}$ define the function $f : \mathbb{N} \to \mathbb{R}$. Since $\{b_n\}$ is a subsequence of $\{a_n\}$ we know that there is a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ so that $b_n = f(g(n))$. Of course we also have $b_{n+1} = f(g(n + 1))$. Now since g is strictly increasing we know that $g(n + 1) > g(n)$, also since f is increasing, we can use Proposition 1.3.7 to conclude that $f(m) \geq f(n)$ whenever $m \geq n$. Substituting $g(n + 1)$ in for m and $g(n)$ in for n, we see that $f(g(n + 1)) \geq f(g(n))$ for all $n \in \mathbb{N}$. That is, $b_{n+1} \geq b_n$ for all $n \in \mathbb{N}$.
$$\square$$

Eventual Properties: (Thanks to Ed Azoff for suggesting the idea of this section.)

   In this section we have looked at some properties of sequences that will be very important in the upcoming sections of this chapter. Actually, in

what follows we are mostly interested in what happens with sequences in the long run, i.e. when n is large. For this reason, it is useful to weaken these definitions to require that they only happen *eventually*. In general, if $\{a_n\}$ is a sequence, we can say that a property of that sequence happens *eventually* if there exists some natural number $N \in \mathbb{N}$ so that the property happens for all $n > N$. For example, we can define

Definition 1.3.17
The sequence $\{a_n\}$ is *eventually increasing* if there exists an $N \in \mathbb{N}$ such that $a_{n+1} \geq a_n$ for all $n > N$.

Similarly, you can define eventually decreasing, or eventually strictly increasing or decreasing sequences. We will leave the details to you.

Example 1.3.18
Let $\{a_n\}$ be the sequence defined by $a_n = n^2 - 8n + 12$, for $n \in \mathbb{N}$. We claim that this sequence is eventually increasing. To prove this, let's examine $a_{n+1} - a_n$ to see if it is eventually positive. We have

$$\begin{aligned} a_{n+1} - a_n &= (n+1)^2 - 8(n+1) + 12 - n^2 + 8n - 12 \\ &= n^2 + 2n + 1 - 8n - 8 + 12 - n^2 + 8n - 12 \\ &= 2n - 7 \end{aligned}$$

so we see that as long as $n > 4$ we have that $a_{n+1} - a_n > 0$. Thus this sequence is eventually increasing.

Notice that the sequence is **not** increasing since $a_1 = 5$ and $a_2 = 0$ so the sequence starts off decreasing.

When dealing with eventually increasing sequences in the abstract, we have to come to terms with the fact that we may not know N explicitly, we just know it exists. Here is an example of how that works:

Proposition 1.3.19
Suppose the sequences $\{a_n\}$ and $\{b_n\}$ are both eventually increasing. We claim that the sequence $\{a_n + b_n\}$ is also eventually increasing.

Proof: Since $\{a_n\}$ and $\{b_n\}$ are both eventually increasing, we know that there are natural numbers N_1 and N_2 so that $a_{n+1} \geq a_n$ for all $n > N_1$ and that $b_{n+1} \geq b_n$ for all $n > N_2$. (**Note that we have to use subscripts on the N's since they are not likely to be the same.**) Now, if we let

N be the greater of N_1 and N_2, i.e. $N = \max\{N_1, N_2\}$, then we know that $a_{n+1} \geq a_n$ and $b_{n+1} \geq b_n$ for all $n > N$ since if $n > N$ it follows that $n > N_1$ and $n > N_2$. Combining these inequalities, we have that $a_{n+1} + b_{n+1} \geq a_n + b_n$ for all $n > N$. Thus the sequence $\{a_n + b_n\}$ is eventually increasing. $\quad\square$

In exercises 21 to 27 you will get a chance to explore other eventual properties of sequences.

Visualizations:

Before ending this section, we mention that there is another way of visualizing a sequence. Essentially this amounts to depicting the set of values of the corresponding function. However, if we merely show the set of values, then we have suppressed a great deal of the information in the sequence, namely, the ordering of the points. As a compromise, to retain this information one often indicates the ordering of the points by labeling a few of them. Of course, just as with a graph, any such picture will only show finitely many of the points from the sequence, but still it may give some understanding of the long term behavior of the sequence.

Example 1.3.20

Consider the sequence given by $a_n = 1/n$. A graph of the first five terms of this sequence was given in *figure* 1.3.1. The first five values of this sequence are given by the projection of this graph to the y-axis as in *figure* 1.3.2.
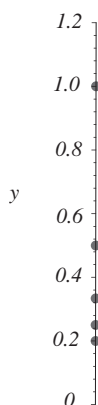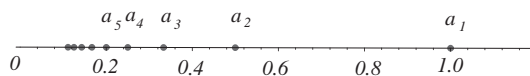

figure 1.3.2

Actually, it is customary to rotate this picture to the horizontal as in *figure* 1.3.3.
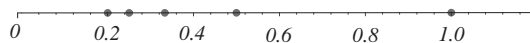
figure 1.3.3

To give an even better indication of the full sequence, we label these first five points and then indicate the location of a few more points as in *figure* 1.3.4.

figure 1.3.4

Review of the Properties of Inequalities and Absolute Values

In the next section we will be working extensively with inequalities and absolute values so here recall some of the basic properties of these.

Properties of Inequalities

Let x, y, and z be real numbers. Then the following are true:

i.) if $x < y$ and $y < z$, then $x < z$
ii.) if $x < y$, then $x + z < y + z$
iii.) if $x < y$ and $z > 0$, then $xz < yz$
iv.) if $x < y$ and $z < 0$, then $xz > yz$
v.) if $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

It is also useful to recall that if x is a real number then exactly one of the following is true: either $x > 0$, or $x < 0$, or $x = 0$. This is called the property of *trichotomy*.

If x is a real number, we define the *absolute value* of x, $|x|$ by

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Properties of Absolute Values

Let $x, y \in \mathbb{R}$, then

a.) $|xy| = |x||y|$
b.) $-|x| \le x \le |x|$
c.) $|x| \le y$ if and only if $-y \le x \le y$
d.) $|x + y| \le |x| + |y|$.

Property d.) is called the *triangle inequality*.

EXERCISES 1.3

1. From the properties of inequalities prove that if $a > b > 0$ then $a^2 > b^2$.

2. From the properties of inequalities prove that if $a > 0, b > 0$ and $a^2 > b^2$, then $a > b$.

3. For the following statements, either prove using the properties of inequalities, or provide a counterexample:

   a.) If $a < b$ and $c < d$, then $ac < bd$.

   b.) If $a < b$ and $0 < c < d$, then $ac < bd$.

   c.) If $0 < a < b$ and $0 < c < d$, then $ac < bd$.

   d.) If $0 < a < b$ and $c < d < 0$, then $ac < bd$.

4. For the following statements, either prove using the properties of inequalities, or provide a counterexample:

   a.) If $a < b$ and $c < d$, then $a/c < b/d$.

   b.) If $a < b$ and $c > d$, then $a/c < b/d$.

   c.) If $a < b$ and $c > d > 0$, then $a/c < b/d$.

   d.) If $0 < a < b$ and $c > d > 0$, then $a/c < b/d$.

5. From the properties of inequalities prove that if $a < b < 0$, then $1/b < 1/a$.

6. a.) Using the definition of the absolute value, show that if $\epsilon > 0$ and $|a| < \epsilon$, then $-\epsilon < a$ and $a < \epsilon$. (This is written more compactly as $-\epsilon < a < \epsilon$.)

   b.) Using the definition of the absolute value, show that if $\epsilon > 0$ and $|a - b| < \epsilon$, then $b - \epsilon < a$ and $a < b + \epsilon$. (Or more compactly, $b - \epsilon < a < b + \epsilon$.)

7. From the properties of absolute values prove that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

8. Which of the following sequences are increasing, strictly increasing, decreasing, strictly decreasing, or none of the above? Justify your answers.

a.) $a_n = n^2 - n, \quad n \in \mathbb{N}$

b.) $c_n = \frac{1}{n+1}, \quad n \in \mathbb{N}$

c.) $b_n = \frac{(-1)^n}{n^2}, \quad n \in \mathbb{N}$

d.) $a_{n+1} = a_n + \frac{1}{n}, \quad$ for $n > 1$, and $\quad a_1 = 1$

e.) $b_n = 1$, for all $n \in \mathbb{N}$.

9. Which of the above sequences are bounded above, or bounded below; which are bounded? Give an upper bound and/or a lower bound when applicable.

10. Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n+1}$. Give explicit formulae for the sequences $c_n = a_n - b_n$ and $d_n = (a_n)(b_n)$. Write the first four terms in each sequence.

11. Let $f(n) = (n+1)(n+2)$ and $g(n) = 2n - 1$. Give an explicit formula for the sequence defined by $f \circ g$. Write out the first 6 terms of the sequence defined by f and that defined by $f \circ g$.

12. Let $a_n = 2n$ for all $n \in \mathbb{N}$ and let $b_n = 2^n$ for all $n \in \mathbb{N}$. Show that b_n is a subsequence of a_n by writing it explicitly as a composition of functions.

13. Prove that every subsequence of a bounded sequence is bounded.

14. Give an example of two sequences $\{a_n\}$ and $\{b_n\}$ neither of which are bounded, but whose sum is bounded.

15. Prove that the sequence $\{|a_n|\}$ is bounded if and only if the sequence $\{a_n\}$ is bounded.

16. Let $g : \mathbb{N} \to \mathbb{N}$ be strictly increasing. Prove by induction that $g(n) \geq n$ for all $n \in \mathbb{N}$.

17. a.) Write a statement similar to Proposition 1.3.7 about decreasing sequences. (Be careful, only one inequality changes.)

    b.) Prove your statement from part a.)

18. a.) Write a statement similar to Proposition 1.3.16 about decreasing sequences.

b.) Prove your statement from part a.)

19. Let a_n be the sequence given recursively by $a_{n+1} = \frac{3a_n+2}{a_n+2}$, with $a_1 = 3$.

   a.) Prove by induction that $a_n > 2$ for all $n \in \mathbb{N}$.

   b.) Prove that the sequence is decreasing.

   c.) What happens to the sequence if we start with $a_1 = 1$?

20. Let a_n be the sequence given recursively by $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$ with $a_1 = 2$.

   a.) Prove by induction that $a_n > 0$ for all $n \in \mathbb{N}$.

   b.) Prove that $a_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$.

   c.) Prove that $\{a_n\}$ is a decreasing sequence.

21. Show that the sequence $\{a_n\}$ given by $a_n = n^2 - 7n + 11$ is eventually increasing.

22. Show that the sequence $\{a_n\}$ given by $a_n = n^3 - 5n^2 - 17n + 21$ is eventually increasing.

23. a.) Make a definition for a sequence to be *eventually positive*.

   b.) Prove that the sum of two eventually positive sequences is eventually positive.

   c.) Prove that the product of two eventually positive sequences is eventually positive.

24. Give an example of two sequences $\{a_n\}$ and $\{b_n\}$ neither of which are eventually positive, but whose sum is eventually positive.

25. Give an example of two sequences $\{a_n\}$ and $\{b_n\}$ neither of which are eventually increasing, but whose sum is eventually increasing.

26. Give an example to show that it is not necessarily true that the product of two eventually increasing sequences is eventually increasing.

27. a.) Make a definition for a sequence to be *eventually bounded*.

   b.) Prove that an eventually bounded sequence is bounded.

1.4 Sequences of Approximations: Convergence

In Calculus we learn *Newton's method* for approximating zeroes of a differentiable function, $f : \mathbb{R} \to \mathbb{R}$. The method recursively defines a sequence of numbers, $\{x_n\}$, which (hopefully) eventually give an arbitrarily good approximation to a point where the graph of the function touches the x-axis.
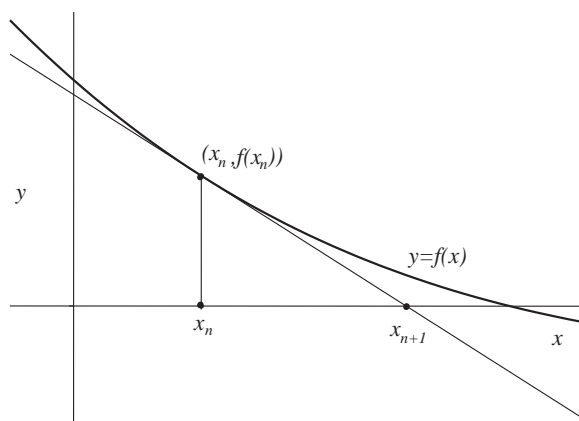

figure 1.4.1

Here is how the method works: First, since this will be a recursive formula, we have to pick an initial value x_1. Our choice here will often make use of some knowledge of the function and the region of its graph in which we expect it to touch the x-axis. Next, we derive the recursive formula for finding x_{n+1} in terms of x_n. We begin by writing out the equation for the tangent line to our function at the point $(x_n, f(x_n))$ on its graph. Since the slope of the tangent line at this point is given by $f'(x_n)$ and the line passes through the point $(x_n, f(x_n))$, we see that this equation is given by:

$$y = f(x_n) + f'(x_n)(x - x_n).$$

Finally, the next point in the sequence is determined by where this line crosses the x-axis (see *figure* 1.4.1). Thus,

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n).$$

Solving for x_{n+1} yields:

$$x_{n+1} = x_n - f(x_n)/f'(x_n).$$

(The sequence stops if at some point $f'(x_n) = 0$, in which case the tangent line is horizontal.)

Example 1.4.1

Let's take $f(x) = x^2 - 2$. Since this function vanishes at $\sqrt{2}$ and $-\sqrt{2}$ we hope that by choosing the correct x_1 we will be able to produce a sequence that gives better and better approximations to $\sqrt{2}$. Since $f'(x) = 2x$, the above recursion formula yields

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = 1/2(x_n + \frac{2}{x_n}).$$

If we make the choice $x_1 = 2$ this gives the sequence explored in exercise 1.3.20.

The rest of this section will be devoted to making precise the notion that this sequence of numbers "eventually gives an arbitrarily good approximation" to the zero of the function. The basic geometric idea is what we will call an *"ϵ-neighborhood"* where ϵ (the Greek letter *epsilon*) simply denotes a positive real number, but will be most interesting when it represents a *small* positive real number. With this in mind, the ϵ-neighborhood around a real number L is just the interval of real numbers given by

$$\{x \in \mathbb{R} \mid L - \epsilon < x < L + \epsilon\} = (L - \epsilon, L + \epsilon).$$

Another important way of describing this same set is given by

$$\{x \in \mathbb{R} \mid |x - L| < \epsilon\}.$$

The ϵ-neighborhood around $L = 2$ with $\epsilon = .1$ is shown in *figure* 1.4.2
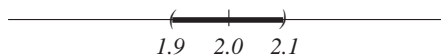
figure 1.4.2

Comments on interval notation:

An *interval* of real numbers is the set of all real numbers between two specified real numbers, say a and b. The two specified numbers, a and b, are called the *endpoints* of the interval. One, both, or neither of the endpoints may be contained in the interval. Since intervals are commonly used in many areas of mathematics, it gets cumbersome to write out the full set notation each time we refer to an interval, so intervals have been given their own special shorthand notation. The following equations should be thought of as defining the notation on the left side of the equality:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$
$$[a, b) = \{x \in \mathbb{R} \mid a \le x < b\}$$
$$(a, b] = \{x \in \mathbb{R} \mid a < x \le b\}$$
$$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}.$$

Now, the size of the ϵ-neighborhood around L is determined by ϵ. The neighborhood gets smaller if the number ϵ gets smaller. Our idea of some number a giving a *good approximation* to the number L is that a should be in some ϵ-neighborhood around L for some small ϵ. Our idea that a sequence *"eventually gives an arbitrarily good approximation"* to L should be that no matter how small we choose ϵ we have that "eventually" all of the elements of the sequence are in that ϵ-neighborhood. This is made mathematically precise in the following definition:

Definition 1.4.2

The sequence $\{a_n\}$ is said to *converge to the real number L* if the following property holds: For every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ so that $|a_n - L| < \epsilon$ for every $n > N$.

If the sequence $\{a_n\}$ converges to L, we write

$$\lim_{n \to \infty} a_n = L.$$

Remark: We will often simply write $\lim a_n = L$ since it is understood that $n \to \infty$.

Definition 1.4.3

A sequence $\{a_n\}$ is said to be *convergent* if there is some real number L, so that $\lim_{n \to \infty} a_n = L$. If a sequence is not convergent it is called *divergent*.

Proposition 1.4.4

a.) The constant sequence $\{a_n\}$, given by $a_n = c$ for all $n \in \mathbb{N}$ converges to c.

b.) The sequence $\{a_n\}$ given by $a_n = \frac{1}{n}$ converges to 0.

Proof:

a.) Since for this sequence we have $|a_n - c| = 0$ for all n, it follows that no matter what ϵ is chosen, we can take N to be 1.

b.) To prove this we will need to use the fact that the sequence of natural numbers is not bounded above (see Example 1.3.10). I.e.,

Given any positive real number r, there is a natural number n, so that $n > r$.

Now suppose we are given some $\epsilon > 0$. Using the above property, find $N \in \mathbb{N}$ with $N > 1/\epsilon$. Then for every $n > N$ we have $n > 1/\epsilon$ also. But this is equivalent to $1/n < \epsilon$, i.e. $|a_n - 0| < \epsilon$.

$\square$

Since the sequence given by $a_n = 1/n$ converges to zero, we know that eventually all of the terms in the sequence are within some small neighborhood of 0. Since any other real number $L \neq 0$ is a finite distance from 0, it seems impossible that the a_n's could also eventually be close to L. We show this for $L = .01$ in the next example

Example 1.4.5

The sequence $\{a_n\}$ given by $a_n = \frac{1}{n}$ does not converge to 0.01.

Proof:

To prove that a sequence $\{a_n\}$ does not converge to L we must show that the condition in the definition fails. Thus we must find just one $\epsilon > 0$ so that for every $N \in \mathbb{N}$ there is an $n > N$ with $|a_n - L| \geq \epsilon$.

In this case we have $L = .01$, let's take $\epsilon = .001$ (actually any positive number less than .01 would work). Now, if $n > 200$ we have $1/n < .005$ and so $|a_n - .01| > .01 - .005 = .005 > .001$. Thus, no matter what N is chosen, we can find n which is larger than both N and 200 and get that $|a_n - .01| \geq \epsilon$. (Notice that in this proof there is lots of flexibility in the choice of ϵ and in the choice of n.)

$\square$

Comments on Negations:

Given some mathematical (or logical) statement, P, the *negation of P* is the new statement that P fails. In the above example the statement P is "the sequence $\{a_n\}$ converges to L", the negation of this statement is "the sequence $\{a_n\}$ does not converge to L". Now, in this case, P is defined by a fairly complicated collection of conditions on $\{a_n\}$ and L, and so to get a useful interpretation of the negation of P, we had to determine what it means for that collection of conditions to fail. Negating a definition can often be a tricky exercise in logic, but it also leads to a better understanding of the definition. After all, understanding how the conditions can fail gives a deeper understanding of when the conditions are satisfied. Thus, it is generally recommended that when reading mathematics, one should stop after each definition and consider the negation of that definition. There are some useful guidelines that can help when writing out a negation of a definition. Here are a two such basic rules:

1.) If a definition requires that two conditions hold, then the negation of the definition will require that at least one of the two conditions fail.

2.) If a definition asks that either condition A or condition B should hold, then then the negation of the definition will require that both of the two conditions must fail.

Actually, these two rules can be made much more general:

1′.) If a definition asks that some condition holds for every element of some set S, then the negation of the definition will require that the condition must fail for at least one element of S.

2′.) If a definition asks that some condition holds for at least one element of some set S, then the negation of the definition will require that the condition must fail for every element of S.

Now the definition that the sequence $\{a_n\}$ limits to L says, "for every $\epsilon > 0$" some condition, let's call it A, holds, thus to negate this definition, we ask that condition A should fail for "at least one $\epsilon > 0$". Now condition A asks that "there exist some $N \in \mathbb{N}$" so that some other condition, let's call it B, should hold. Thus for A to fail we need to show that B fails "for every $N \in \mathbb{N}$". Finally condition B is the statement that "$|a_n - L| < \epsilon$ for every $n > N$", so B fails if "there exists just one $n > N$ so that $|a_n - L| \geq \epsilon$." Putting this string of logic together, we come up with the statement,

The sequence $\{a_n\}$ does not converge to L if there exists an $\epsilon > 0$ such that for every $N \in \mathbb{N}$ there exists an $n > N$ so that $|a_n - L| \geq \epsilon$, just as stated at the beginning of the proof in example 1.4.5. One should note that, in practice, showing that this condition is satisfied is usually accomplished by finding a subsequence $\{b_n\}$ of $\{a_n\}$ so that there is an $\epsilon > 0$ with $|b_n - L| \geq \epsilon$ for all $n \in \mathbb{N}$.

Another important aspect of example 1.4.5 is that it raises the question of *uniqueness* of the limit, i.e. is it possible that a given sequence has two or more distinct limits?

Proposition 1.4.6

Let $\{a_n\}$ be a sequence and L and M real numbers with $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$. Then $L = M$.

Proof:

Let $\epsilon > 0$ be given. Since $\lim a_n = L$ we know that we can find a natural number N_1 so that $|a_n - L| < \epsilon$ for every $n > N_1$. Similarly, since $\lim a_n = M$, we can find a natural number N_2 so that $|a_n - M| < \epsilon$ whenever $n > N_2$. (We use the subscripts on N_1 and N_2 since these numbers may not be the same.) Now let N be the larger of the two numbers N_1 and N_2. Then whenever $n > N$, we have both $n > N_1$ and $n > N_2$, thus, for such n, we have both $|a_n - L| < \epsilon$ and $|a_n - M| < \epsilon$. *Now here is a trick that will come in handy in many arguments about limits.* We look at $|L - M|$, add and subtract a_n, and then use the triangle inequality. So we have, whenever $n > N$,

$$
\begin{aligned}
|L - M| &= |L - a_n + a_n - M| \\
&\leq |L - a_n| + |a_n - M| \\
&< \epsilon + \epsilon \\
&= 2\epsilon.
\end{aligned}
$$

Now this inequality is true no matter what is the original choice of ϵ. But *if a nonnegative number is less than every positive number, it must be zero.* So we conclude that $|L - M| = 0$, i.e. $L = M$.

$\square$

In the next section we will discuss some properties of limits that will enable us to evaluate some limits without having to revert to the definition. However, before going there, it will strengthen our understanding of definition 1.4.2 if we prove convergence of another example directly from the definition.

Example 1.4.7

In this example we will examine a proof that

$$
\lim_{n\to\infty} \frac{2n + 1}{3n - 1} = 2/3.
$$

Before getting involved in the proof, using an arbitrary $\epsilon > 0$, let's practice with a particular ϵ, say $\epsilon = .01$. So, letting $a_n = (2n+1)/(3n-1)$, we want

to find (explicitly) a natural number N so that whenever $n > N$ we have $|a_n - 2/3| < .01$. Now I might impress you with my acute foresight if I tell you right now that I am going to choose $N = 55$ and then prove to you that this N works. Instead, I will be a bit more honest with you and show you how I come up with this value for N.

We want a condition on n that will guarantee that

$$\left| \frac{(2n+1)}{(3n-1)} - \frac{2}{3} \right| < .01.$$

Let's try to find an equivalent inequality from which it is easier to read off a condition on n. First, we simplify the expression inside the absolute values by noting

$$\left| \frac{2n+1}{3n-1} - \frac{2}{3} \right| = \left| \frac{3(2n+1)}{3(3n-1)} - \frac{2(3n-1)}{3(3n-1)} \right|$$
$$= \left| \frac{5}{3(3n-1)} \right|.$$

Next we rewrite the inequality without the absolute values, namely we use the fact that the inequality

$$\left| \frac{5}{3(3n-1)} \right| < .01$$

is equivalent to the two inequalities

$$-.01 < \frac{5}{3(3n-1)} < .01. \tag{$*$}$$

Now we notice that since n is a natural number, it is always true that $3n-1 > 0$, hence $\frac{5}{3(3n-1)} > 0$ as well. Thus, the first inequality is true for all natural numbers n. Multiplying the second inequality by the positive number $3n-1$, and then by 100, yields, $\frac{500}{3} < 3n - 1$. Further simplification leads to the condition

$$n > \frac{503}{9} \approx 55.9.$$

This shows that if we take n to be any natural number greater than 55 we will get our desired inequality, thus we can take $N = 55$ (or any natural number bigger than 55).

Now in general we can go through the above argument for any given particular value of ϵ, or we can go through the algebra leaving the ϵ in. Of course, the N that we get will depend on the choice of ϵ. To see how this works in general, lets go back to equation (*) and replace the .01's with ϵ, thus we have

$$-\epsilon < \frac{5}{3(3n-1)} < \epsilon.$$

Again, the fraction in the middle is positive so the left inequality is trivial, and we begin simplifying the right inequality by multiplying by $3(3n-1)$ to get $5 < 3(3n-1)\epsilon$. Now we divide by ϵ (this corresponds to multiplying by 100 when we took $\epsilon = .01$) to get $5/\epsilon < 9n - 3$. Simplifying this, we see that we want

$$n > \frac{5}{9\epsilon} + \frac{1}{3}.$$

That is, we can take N to be any natural number greater than $\frac{5}{9\epsilon} + \frac{1}{3}$.

Finally, before leaving this example, let's write out the formal proof that $\lim \frac{2n+1}{3n-1} = \frac{2}{3}$.

Proof: Let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ in such a way that $N > \frac{5}{9\epsilon} + \frac{1}{3}$. then if $n > N$ we have

$$n > \frac{5}{9\epsilon} + \frac{1}{3},$$

which is equivalent to

$$3n - 1 > \frac{5}{3\epsilon}.$$

Since n is a natural number, we know that $3n-1 > 0$, so the above inequality is equivalent to

$$\frac{5}{3(3n-1)} < \epsilon.$$

Also, since the left hand side is positive, we have trivially that

$$-\epsilon < \frac{5}{3(3n-1)} < \epsilon.$$

Now a short computation shows that

$$\frac{5}{3(3n-1)} = \frac{3(2n+1) - 2(3n-1)}{3(3n-1)} = \frac{2n+1}{3n-1} - \frac{2}{3},$$

so our inequalities give

$$-\epsilon < \frac{2n+1}{3n-1} - \frac{2}{3} < \epsilon,$$

i.e.

$$\left| \frac{2n+1}{3n-1} - \frac{2}{3} \right| < \epsilon.$$

Thus we have shown that if $\epsilon > 0$ is given, and we choose a natural number $N > \frac{5}{6\epsilon}$, then whenever $n > N$ we have $|a_n - 2/3| < \epsilon$, i.e., $\lim a_n = 2/3$.

$\square$

Alright, that wasn't so bad. Let's try a slightly more complicated example:

Example 1.4.8

Let us prove that

$$\lim_{n\to\infty} \frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} = \frac{2}{5}.$$

To figure out what N should be, we start off like we did in the last example. First, let $\epsilon > 0$ be given. We want to find $N \in \mathbb{N}$ so that

$$-\epsilon < \frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} < \epsilon$$

for all $n > N$. A little algebra shows that this is equivalent to

$$-\epsilon < \frac{11n - 27}{5(5n^2 + 2n + 1)} < \epsilon.$$

Now the fraction in the middle of the above set of inequalities is always positive as long as $n > 2$, so as long as we make sure in the end that $N > 2$, we don't have to worry about the first inequality. On the other hand, solving for n in the second inequality is much more complicated than it was in the previous example. Fortunately, we don't need to find n in terms of ϵ *exactly*, we just need an approximation that will guarantee the inequality we need. Here is how to proceed: First, notice

$$\frac{11n - 27}{5(5n^2 + 2n + 1)} = \frac{n(11 - \frac{27}{n})}{5n^2(5 + \frac{2}{n} + \frac{1}{n^2})}$$

$$= \frac{1}{n}\left(\frac{11 - \frac{27}{n}}{5(5 + \frac{2}{n} + \frac{1}{n^2})} \right).$$

Now, since $n > 0$, it is easy to see that $11 - \frac{27}{n} < 11$ and that $5 + \frac{2}{n} + \frac{1}{n^2} > 5$. This second inequality implies that $\dfrac{1}{5 + \frac{2}{n} + \frac{1}{n^2}} < \dfrac{1}{5}$ (both terms are positive), and so we have that

$$\frac{(11 - \frac{27}{n})}{5(5 + \frac{2}{n} + \frac{1}{n^2})} < \frac{11}{5 \cdot 5} = \frac{11}{25}.$$

Therefore we see that

$$\frac{1}{n}\left(\frac{11 - \frac{27}{n}}{5(5 + \frac{2}{n} + \frac{1}{n^2})}\right) < \frac{11}{25n},$$

and so, if we choose N large enough to guarantee that $\frac{11}{25n} < \epsilon$ we will certainly have that $\dfrac{1}{n}\left(\dfrac{11 - \frac{27}{n}}{5(5 + \frac{2}{n} + \frac{1}{n^2})}\right) < \epsilon$ as well. But this is easy enough, just choose N to be greater than $\frac{11}{25\epsilon}$. Don't forget that we also need to make sure $N > 2$. This is done by just saying that we choose $N \in \mathbb{N}$ so that $N > \max(2, \frac{11}{25\epsilon})$. Ok, now let's put this all together in a formal proof:

Proof: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that $N > \max(2, \frac{11}{25\epsilon})$. If $n > N$, it follows that

$$\frac{11}{25n} < \epsilon.$$

However, since $n > 0$, we know that $11 > 11 - \frac{27}{n}$ and $5 < 5 + \frac{2}{n} + \frac{1}{n^2}$. Hence, we can conclude that

$$\frac{1}{n}\left(\frac{11 - \frac{27}{n}}{5(5 + \frac{2}{n} + \frac{1}{n^2})}\right) < \frac{11}{25n} < \epsilon.$$

Multiply the left hand side of the above inequality by $\frac{n}{n}$ to get

$$\frac{11n - 27}{5(5n^2 + 2n + 1)} < \epsilon.$$

Algebra shows that

$$\frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} = \frac{11n - 27}{5(5n^2 + 2n + 1)},$$

so we have that if $n > N$, then

$$\frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} < \epsilon.$$

Also, since $n > N > 2$, it is clear that

$$\frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} = \frac{11n - 27}{5(5n^2 + 2n + 1)} > 0 > -\epsilon.$$

Thus, for $n > N$, we have shown

$$\left| \frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} - \frac{2}{5} \right| < \epsilon.$$

$\square$

To conclude this section we will discuss a few general properties of convergence that can help to determine convergence or divergence of a sequence. First, it is useful to notice that if we change the first K terms of a sequence it won't affect the convergence of the sequence. We leave the proof of this to the exercises.

Proposition 1.4.9
Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences and that there is a $K \in \mathbb{N}$ so that $a_n = b_n$ for all $n > K$. Assume also that $\{a_n\}$ converges to the real number L, then the sequence $\{b_n\}$ also converges to L.

The next proposition gives an easy first check for convergence since its negation says that if a sequence is not bounded, then it must diverge.

Proposition 1.4.10
Every convergent sequence is bounded.

Proof: Let $\{a_n\}$ be a convergent sequence and let L be its limit. Then we know that for any $\epsilon > 0$ we can find an $N \in \mathbb{N}$ so that $|a_n - L| < \epsilon$ for all $n > N$. In particular, we can take ϵ to be equal to 1, then we know that there is some natural number N_1 so that $|a_n - L| < 1$ for all $n > N_1$. We can now turn this into a bound for $|a_n|$ by noting that $|a_n| = |a_n - L + L| \leq |a_n - L| + |L|$ by the triangle inequality, hence we conclude

$$|a_n| < |L| + 1$$

for all $n > N_1$.

Now, to get a bound on $|a_n|$ for all $n \in \mathbb{N}$, we let M be the maximal element of the set

$$\{|a_1|, |a_2|, ..., |a_{N_1}|, |L| + 1\}.$$

Then certainly $|a_n| \leq M$ when $1 \leq n \leq N_1$, and also $|a_n| < |L| + 1 \leq M$ for $n > N_1$. Thus we have that $|a_n| \leq M$ for all n.

□

Comments on Maxima:

   An element M of a set, S, is called a *maximal element* (or just a *maximun*) if $s \leq M$ for all $s \in S$. Thus, in the above proof, with $S = \{|a_1|, |a_2|, ..., |a_{N_1}|, |L| + 1\}$, we can conclude that each of the elements, $|a_1|, |a_2|, ..., |a_{N_1}|, |L| + 1$ is less than or equal to the maximum, M. The subtlety is that *not every set has a maximal element*. Indeed, any set that is not bounded above (for example $\mathbb{N}$) will have no maximal element. But even sets which are bounded above may not have a maximal element. For example, the set $\{1 - 1/n \mid n \in \mathbb{N}\}$ is bounded above by 1, but has no maximal element. Then why are we allowed in the above proof to simply say, "To get a bound for the entire sequence, we let M be the maximal element of the set $\{|a_1|, |a_2|, ..., |a_{N_1}|, |L| + 1\}$?" The reason we can do this is that *every finite set of real numbers has a maximal element*.

Example 1.4.11

   The sequence given by $a_n = n$ is not bounded (as stated in Example 1.3.10), hence, by the above proposition, it must be divergent.

   The next proposition and its corollary give us another good method for showing that a sequence diverges.

Proposition 1.4.12

   Suppose $\{a_n\}$ is a sequence converging to the real number L and that $\{b_n\}$ is a subsequence of $\{a_n\}$. Then $\{b_n\}$ converges to L also.

Proof: Let $f : \mathbb{N} \to \mathbb{R}$ be the function corresponding to the sequence $\{a_n\}$ and let $g : \mathbb{N} \to \mathbb{N}$ be the the strictly increasing function defining $\{b_n\}$ as a subsequence of $\{a_n\}$, i.e., $b_n = f(g(n))$. Now let $\epsilon > 0$ be given. Since $\lim a_n = L$ we can find $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n > N$. Rewriting this in terms of f we have $|f(n) - L| < \epsilon$ for all $n > N$. But since g is strictly increasing, it follows that $g(n) \geq n$ for all $n \in \mathbb{N}$. Hence if $n > N$ then also $g(n) > N$ and so we can conclude that $|f(g(n)) - L| < \epsilon$ whenever $n > N$. Since $f(g(n)) = b_n$, this is the same as saying $\lim b_n = L$.

□

Corollary 1.4.13

Suppose that $\{a_n\}$ is a sequence with two convergent subsequences, $\{b_n\}$ and $\{c_n\}$. If $\lim_{n\to\infty} c_n \neq \lim_{n\to\infty} b_n$ then $\{a_n\}$ must be divergent.

Example 1.4.14

The sequence given by $a_n = (-1)^n$ has even terms all equal to 1 and odd terms all equal to -1. Thus the subsequence of even terms converges to 1 and the subsequence of odd terms converges to -1. From the corollary, we conclude that the sequence $\{a_n\}$ diverges.

This example provides the last case in the following general result about geometric sequences.

Proposition 1.4.15

Let $r \in \mathbb{R}$ and consider the geometric sequence $\{a_n\}$ given by $a_n = r^n$. Then

a.) if $r = 1$ we have $\lim_{n\to\infty} a_n = 1$

b.) if $|r| < 1$ we have $\lim_{n\to\infty} a_n = 0$

c.) if $|r| > 1$ or $r = -1$ the sequence $\{a_n\}$ diverges.

Proof: We have already proven part a.) in Proposition 1.4.4 since if $r = 1$ then $a_n = 1$ for all $n \in \mathbb{N}$. Also, the case that $r = -1$ is dealt with in example 1.4.14.

For the rest of part c.) let's begin by looking at the case where $r > 1$. Let $c = r - 1$, so $c > 0$ and $r = 1 + c$. A straightforward induction argument shows that $r^n \geq 1 + nc$. Indeed, for the case $n = 1$ we have equality by the definition of c. If we now assume that $r^k \geq 1 + kc$ then $r^{k+1} = rr^k \geq (1 + c)(1 + kc) = 1 + (k + 1)c + kc^2 \geq 1 + (k + 1)c$. Thus we have that $r^{k+1} \geq 1 + (k + 1)c$ completing the induction step. Finally, we remark that the sequence given by $1 + nc$ is unbounded since if $M \in \mathbb{R}$ we can find $n \in \mathbb{N}$ so that $n > (M - 1)/c$, from which it follows that $1 + nc > M$. Since the sequence $1 + nc$ is unbounded, and we know that $r^n > 1 + nc$, it follows that the sequence r^n is unbounded, and thus divergent. For the case that $r < -1$, we simply notice that $|r^n| = |r|^n$, and since $|r| > 1$ we now know that the sequence $|r|^n$ is unbounded. But since the sequence $|r^n|$ is unbounded it follows that the sequence r^n is unbounded (see exercise 1.3.15).

Now we are ready for part b.). First, since $|r| < 1$ we know that $\frac{1}{|r|} > 1$ so, as above, we can write $\frac{1}{|r|} = 1 + c$ for some $c > 0$. The above induction argument shows that $(\frac{1}{|r|})^n \geq 1 + nc$ and so, by inverting, we have that

$|r^n| = |r|^n \le \frac{1}{1+nc}$. Now let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ so that $N > (\frac{1}{\epsilon} - 1)/c$. Then if $n > N$ we have that $n > (\frac{1}{\epsilon} - 1)/c$. This is equivalent to $1 + nc > \frac{1}{\epsilon}$, which, in turn, implies that $\frac{1}{1+nc} < \epsilon$. Thus we see that if $n > N$, we have

$$|r^n - 0| = |r|^n \le \frac{1}{1+nc} < \epsilon,$$

showing that $\lim r^n = 0$.

□

In proposition 1.4.10 we proved that every convergent sequence is bounded. We will need a few further results along these lines for section 1.6. We state them here and leave the proofs for you to do as exercises.

Proposition 1.4.16

Let $\{a_n\}$ be a convergent sequence with $\lim a_n = \alpha$ and assume that U is an upper bound for $\{a_n\}$, i.e. $a_n \le U$ for all $n \in \mathbb{N}$. Then $\alpha \le U$.

Proof Hint: Suppose on the contrary that $\alpha > U$ and let $\epsilon = (\alpha - U)/2$. Use the definition of the limit to show how to find a particular $N \in \mathbb{N}$ with $a_N > U$. This contradicts the assumption. (See *figure* 1.4.3.)
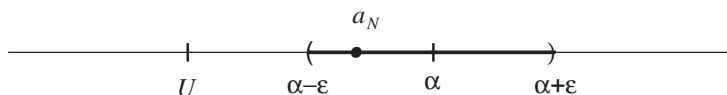
figure 1.4.3

**Remark:** Of course there is a similar proposition for lower bounds.

The next proposition says that if $\{a_n\}$ is an *increasing* convergent sequence, then the limit, $\lim a_n$, gives an upper bound for $\{a_n\}$. Of course, there is a similar proposition for decreasing sequences.

Proposition 1.4.17

Let $\{a_n\}$ be an increasing sequence with $\lim a_n = \alpha$, then $a_n \le \alpha$ for all $n \in \mathbb{N}$.

Proof Hint: Suppose on the contrary that there is some particular a_k with $a_k > \alpha$. Let $\epsilon = (a_k - \alpha)/2$ and use the definition of the limit to show that

there is some $m > k$ with $a_m < a_k$, contradicting the fact that the sequence is increasing. (See *figure* 1.4.4.)
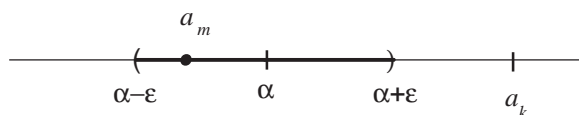
figure 1.4.4

Remark: Combining these two propositions for the case of an increasing convergent sequence $\{a_n\}$ we see that $\lim a_n$ is an upper bound for this sequence and that it is less than or equal to all upper bounds for this sequence, i.e. it is the least of all of the upper bounds. Thus it is called the *least upper bound* of the sequence $\{a_n\}$. Similarly, one can prove that if $\{b_n\}$ is a decreasing convergent sequence, then $\lim b_n$ gives the *greatest lower bound* of $\{b_n\}$.

EXERCISES 1.4

1. Let $a_n = \frac{2n+4}{3n-2}$. Find $N \in \mathbb{N}$ so that $|a_n - 2/3| < .01$ for all $n > N$. Justify your work.

2. Let $a_n = \frac{3n+2}{2n-15}$. Find $N \in \mathbb{N}$ so that $|a_n - 3/2| < .05$ for all $n > N$. Justify your work.

3. Let $a_n = \frac{n^2+2}{5n^2+1}$. Find $N \in \mathbb{N}$ so that $|a_n - 1/5| < .02$ for all $n > N$. Justify your work.

4. Let $a_n = 3 - 1/n$. Using the definition of the limit, prove that $\lim_{n\to\infty} a_n = 3$.

5. Let $a_n = \frac{2n+4}{3n+1}$. Using the definition of the limit, prove that $\lim_{n\to\infty} a_n = 2/3$.

6. Let $a_n = \frac{5-n}{3n-7}$. Using the definition of the limit, prove that $\lim_{n\to\infty} a_n = -1/3$.

7. Let $a_n = \frac{5n^2+4}{3n^2+4}$. Using the definition of the limit, prove that $\lim_{n\to\infty} a_n = 5/3$.

8. Let $a_n = \frac{2n^2+4n-3}{3n^2+2n+1}$. Using the definition of the limit, prove that $\lim_{n\to\infty} a_n = 2/3$.

9. Let $a_n = 1/n$. Using the definition of the limit, prove that a_n does not converge to $1/4$.

10. Eddy wrote on his midterm exam that the definition of the limit is the following: The sequence $\{a_n\}$ converges to the real number L if there exists an $N \in \mathbb{N}$ so that for every $\epsilon > 0$ we have $|a_n - L| < \epsilon$ for all $n > N$. Show Eddy why he is wrong by demonstrating that if this were the definition of the limit then it would not be true that $\lim_{n\to\infty} \frac{1}{n} = 0$. (Hint: What does it mean if $|a - b| < \epsilon$ for every $\epsilon > 0$?)

11. Sally wrote on her midterm exam that the definition of the limit is the following: The sequence $\{a_n\}$ converges to the real number L if for every $\epsilon > 0$ we have $|a_n - L| < \epsilon$ for all $n \in \mathbb{N}$. Show Sally why she is wrong by demonstrating that if this were the definition of the limit then it would not be true that $\lim_{n\to\infty} \frac{1}{n} = 0$.

12. Give the explicit conditions for a sequence to diverge, i.e., give a "positive" version of the negation of the definition of convergence (definition 1.4.2 and 1.4.3).

13. Explicitly write out the negation of the incorrect definition given by Eddy in exercise 10.

14. Explicitly write out the negation of the incorrect definition given by Sally in exercise 11.

15. Prove Proposition 1.4.9.

16. Prove that $\lim |a_n| = 0$ if and only if $\lim a_n = 0$.

17. Suppose that $\lim_{n\to\infty} a_n = L$.

    a.) Prove that if $a_n \geq 0$ for all $n \in \mathbb{N}$, then $L \geq 0$.

b.) Give an example to show that $a_n > 0$ for all $n \in \mathbb{N}$ does not imply that $L > 0$.

18. Supose that $\lim_{n\to\infty} a_n = L$ and that $r < L$. Show that there exists an $N \in \mathbb{N}$ such that $a_n > r$ for all $n > N$ (i.e. a_n is eventually greater than r).

19. Suppose that $\lim_{n\to\infty} a_n = L$ and that r is a real number. Prove that $\lim_{n\to\infty} r a_n = rL$.

20. Prove from the definition of divergence that the sequence $a_n = (-1)^n$ is divergent.

21. Evaluate the following limits.

a.) $\lim (3/2)^n$

b.) $\lim (1/2)^n$

22. Provide the details of the proof for Proposition 1.4.16.

23. Provide the details of the proof for Proposition 1.4.17.

1.5 Tools for Computing Limits

In this section we will develop a number of tools that will allow us to compute some limits of sequences without going back to the definition.

Proposition 1.5.1

Suppose $\lim a_n = 0$ and that $\{b_n\}$ is a bounded sequence. Then we have $\lim a_n b_n = 0$.

Proof: Since $\{b_n\}$ is bounded there is some number $M > 0$ with $|b_n| < M$ for all $n \in \mathbb{N}$. Also, since $\lim a_n = 0$, when $\epsilon > 0$ is given we can find an $N \in \mathbb{N}$ so that $|a_n| < \epsilon/M$ for all $n > N$. But then

$$|a_n b_n| = |a_n||b_n| < \frac{\epsilon}{M} M = \epsilon$$

whenever $n > N$.

$\square$

Example 1.5.2

The above proposition can be applied, using $a_n = 1/n$, to show that the following limits are zero.

a.) $\lim_{n\to\infty} \frac{1}{n^2} = 0$

b.) $\lim_{n\to\infty} \frac{1}{n^k} = 0$, for any $k \in \mathbb{N}$

c.) $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$

Remark:

From the definition of the limit we know that if $\lim a_n = L$ then when $\epsilon > 0$ is given, we can find an $N \in \mathbb{N}$ so that $|a_n - L| < \epsilon$ for all $n > N$. In the above proof we have made a slight variation on this sentence without explanation. Here is what is going on: if some $\epsilon > 0$ is given to us, and M is some fixed, positive number, then we can think of the quotient ϵ/M as a "new ϵ", let's call it $\tilde{\epsilon}$, that is, let $\tilde{\epsilon} = \epsilon/M$. Then, since $\lim a_n = L$, there is some number $\tilde{N} \in \mathbb{N}$ so that $|a_n - L| < \tilde{\epsilon}$ for all $n > \tilde{N}$. We will use this trick over and over again in the proof of the next proposition.

The next theorem allows us to compute new limits by algebraically manipulating limits which we already understand.

Proposition 1.5.3

Suppose that $\lim a_n = L$ and $\lim b_n = M$. Then

a.) $\lim(a_n + b_n) = L + M$

b.) $\lim(a_n b_n) = LM$

c.) If $M \neq 0$ then $\lim \frac{a_n}{b_n} = \frac{L}{M}$.

Proof: We will only give a sketchy proof for part a.); we leave the details for you to prove in the exercises. For parts b.) and c.) we will try to be more complete.

For part a.) use that

$$|(a_n + b_n) - (L + M)| = |(a_n - L) + (b_n - M)|$$
$$\leq |a_n - L| + |b_n - M|.$$

Then if you want to find N large enough so that $|(a_n+b_n)-(L+M)| < \epsilon$ for $n > N$, it is enough to find it large enough so that $|a_n - L| < \epsilon/2$ and $|b_n - M| < \epsilon/2$. But there is an N_1 so that the first inequality holds for $n > N_1$ and an N_2 so that the second one holds for $n > N_2$, so they both must hold if we take $n > N$ where $N = \max\{N_1, N_2\}$.

For part b.), in order to use the assumptions that $\lim a_n = L$ and $\lim b_n = M$ we need to try to separate out terms like $|a_n - L|$ and $|b_n - M|$ from the expression $|a_n b_n - LM|$ which needs to be studied when proving $\lim a_n b_n = LM$. To do this we use the old algebraic trick of adding and subtracting the same thing, namely $|a_n b_n - LM| = |a_n b_n - L b_n + L b_n - LM|$. This, in turn yields

$$
\begin{aligned}
|a_n b_n - LM| &= |a_n b_n - L b_n + L b_n - LM| \\
&\leq |a_n b_n - L b_n| + |L b_n - LM| \\
&\leq |a_n - L||b_n| + |L||b_n - M|.
\end{aligned}
$$

Now, if $\epsilon > 0$ is given, we know we can pick $N_1 \in \mathbb{N}$ so that $|b_n - M| < \epsilon/2|L|$ for $n > N_1$, so the second term above is no problem. For the first term, we have to recall that since the sequence $\{b_n\}$ is convergent, we know it is bounded. Hence, there is a number $B > 0$ such that $|b_n| < B$ for all $n \in \mathbb{N}$, so the first term above (i.e. $|a_n - L||b_n|$) is less than $|a_n - L|B$ for all n. Now we can pick $N_2 \in \mathbb{N}$ so that $|a_n - L| < \epsilon/2B$ for all $n > N_2$. Hence, if we let $N = \max\{N_1, N_2\}$, then whenever $n > N$ we have both

$$
|a_n - L||b_n| < \frac{\epsilon}{2B} B = \frac{\epsilon}{2}
$$

and

$$
|L||b_n - M| < \frac{\epsilon}{2|L|}|L| = \frac{\epsilon}{2}.
$$

Putting these inequalities together we see that when $n > N$ we have

$$
|a_n b_n - LM| \leq |a_n - L||b_n| + |L||b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
$$

Before proceeding to part c.) we need to confess that the statement doesn't actually make sense! The problem is that although we assume that $M \neq 0$ it is entirely possible that some of the b_n's are zero, and thus the sequence $\{a_n/b_n\}$ may not make sense. To understand that this is really not a problem we have the following lemma which tells us that we can be sure that, at least eventually, the b_n's are nonzero, and thus, since the beginning of a sequence does not affect its limit, we can ignore those first terms where some of the b_n's may be zero.

Lemma 1.5.4

If $\lim a_n = L$ and $L > 0$ then there is a natural number N so that $a_n > L/2$ for all $n > N$.

We leave the proof of the lemma to the exercises, but here is a hint: Choose $\epsilon = L/2$ and look at *figure* 1.5.1 below.

figure 1.5.1

Now, for the proof of part c.), we begin by looking at the special case that $a_n = 1$ for all n, and that $M > 0$ (the argument for $M < 0$ is similar). That is, we will start by showing that if $\lim b_n = M$ and $M > 0$, then $\lim 1/b_n = 1/M$. Applying the above lemma to the sequence $\{b_n\}$ we choose $N_1 \in \mathbb{N}$ so that $b_n > M/2$ for all $n > N_1$. Using the properties of inequalities, we see that this implies that $1/|b_n| < 2/M$ for all $n > N_1$. Then we see that, when $n > N_1$, we have

$$\left| \frac{1}{b_n} - \frac{1}{M} \right| = \left| \frac{M - b_n}{b_n M} \right|$$
$$= \left| \frac{1}{b_n} \right| \left| \frac{1}{M} \right| |M - b_n|$$
$$< \frac{2}{M} \frac{1}{M} |M - b_n|$$
$$= \frac{2}{M^2} |M - b_n|.$$

Now, if $\epsilon > 0$ is given to us, we can find an $N_2 \in \mathbb{N}$ so that $|b_n - M| < M^2 \epsilon/2$. Then, if we take N to be the maximum of N_1 and N_2, we have that for $n > N$,

$$\left| \frac{1}{b_n} - \frac{1}{M} \right| < \frac{2}{M^2} |M - b_n|$$
$$< \frac{2}{M^2} \frac{M^2 \epsilon}{2}$$
$$= \epsilon.$$

To prove the statement with a general sequence $\{a_n\}$ we simply apply part b.) to the two sequences $\{a_n\}$ and $\{1/b_n\}$, using the fact that we now know that $\lim 1/b_n = 1/M$.

$\square$

Example 1.5.5

An easy application of part b.) of Proposition 1.5.3 yields the following result:

> *If* $\lim a_n = L$ *and* r *is a real number, then* $\lim r a_n = rL$.

To see how this follows from part b.) above, just let $\{b_n\}$ be the sequence given by $b_n = r$ for all n.

Example 1.5.6

Proposition 1.5.3 also helps us to evaluate limits of sequences of rational expressions of n. For example if $\{a_n\}$ is given by

$$a_n = \frac{3n^2 + 2n - 2}{2n^2 + 3n + 5}$$

we can rewrite this expression (by dividing top and bottom by n^2) to get

$$a_n = \frac{3 + \frac{2}{n} - \frac{2}{n^2}}{2 + \frac{3}{n} + \frac{5}{n^2}}.$$

Applying parts a.) and b.) of Proposition 1.5.3 repeatedly (and using example 1.5.5) we see that

$$\lim(3 + \frac{2}{n} - \frac{2}{n^2}) = 3$$

and

$$\lim(2 + \frac{3}{n} + \frac{5}{n^2}) = 2,$$

and so, by applying part c.) of Proposition 1.5.3, we have

$$\lim a_n = \frac{3}{2}.$$

Another result that can be useful in conjunction with the above Proposition says that if a sequence behaves in the limit like a convergent geometric sequence, then it too must converge:

Proposition 1.5.7
Let $\{a_n\}$ be a sequence satisfying $\lim \dfrac{|a_{n+1}|}{|a_n|} = r$, with $r < 1$. Then $\lim a_n = 0$.

Proof: Since $r < 1$, we can find a real number c with $r < c < 1$ (e.g. $c = (r+1)/2$). Now let $\epsilon = c - r$ (notice that $\epsilon > 0$) and choose $N \in \mathbb{N}$ so that

$$\left| \frac{|a_{n+1}|}{|a_n|} - r \right| < \epsilon$$

whenever $n > N$. Then, in particular, we have

$$\frac{|a_{n+1}|}{|a_n|} < r + \epsilon = r + (c - r) = c$$

whenever $n > N$. Rewriting this , we have $|a_{n+1}| < c|a_n|$, when $n > N$.

Now, since the existence (and value) of a limit does not depend on the first N terms of the sequence, we can assume that $|a_{n+1}| < c|a_n|$ for **all** $n \in \mathbb{N}$. A simple induction argument then leads to the conclusion that $|a_{n+1}| < c^n|a_1|$ for all $n \in \mathbb{N}$. The sequence $\{c^n\}$ is geometric with $0 < c < 1$, so we know that $\lim c^n|a_1| = 0$. Thus, given $\epsilon > 0$, we can find $N \in \mathbb{N}$ so that $|c^n a_1| < \epsilon$ for all $n > N$. But since $|a_{n+1}| < |c^n a_1|$, we conclude that $|a_{n+1}| < \epsilon$ for all $n > N$. Thus $\lim a_n = 0$. $\qquad\square$

Example 1.5.8
Consider the sequence given by $a_n = (-1)^n \dfrac{n}{2^n}$ for $n \in \mathbb{N}$. We have

$$\frac{|a_{n+1}|}{|a_n|} = \left(\frac{n+1}{2^{n+1}} \right) \left(\frac{2^n}{n} \right) = \frac{n+1}{2n}.$$

Using Proposition 1.5.3 we can compute that $\lim \frac{n+1}{2n} = \frac{1}{2}$, and so we conclude that $\lim a_n = 0$.

For the remainder of this section we will discuss two results from Calculus which help us to evaluate even more complicated limits. The first of these is a theorem about continuous functions. Recall the definition

Definition 1.5.9
   A function $f : \mathbb{R} \to \mathbb{R}$ is called *continuous at* $c \in \mathbb{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - f(c)| < \epsilon,$$

whenever $|x - c| < \delta$.

   The first result is then:

Proposition 1.5.10
   Suppose that $\lim a_n = L$ and that $f(x)$ is continuous at L, then $\lim f(a_n) = f(L)$.

Proof: Given some $\epsilon > 0$, we need to find $N \in \mathbb{N}$ so that $|f(a_n) - f(L)| < \epsilon$ for all $n > N$. Since f is continuous at L we know that there is a $\delta > 0$ so that $|f(x) - f(L)| < \epsilon$ whenever x satisfies $|x - L| < \delta$. This tells us that, if we can find N so that $|a_n - L| < \delta$ for all $n > N$, then we will have the desired result. But the fact that $\lim a_n = L$ tells us exactly this (using δ in place of ϵ in the definition of $\lim a_n$). We can write this proof out formally as follows:

   Let $\epsilon > 0$ be given. Since f is continuous at L, there is some $\delta > 0$ so that $|f(x) - f(L)| < \epsilon$ whenever $|x - L| < \delta$. Then, since $\lim a_n = L$, we can find $N \in \mathbb{N}$ so that for every $n > N$, we have $|a_n - L| < \delta$. In particular, for $n > N$, $|f(a_n) - f(L)| < \epsilon$. $\qquad\square$

Example 1.5.11
   In example 1.5.6 we saw that $\lim a_n = 3/2$ when $a_n = \frac{3n^2 + 2n - 2}{2n^2 + 3n + 5}$. Now if we consider the continuous functions $f(x) = \sqrt{x}$, $g(x) = x^2$, and $h(x) = \ln(|x|)$,

we get

$$\lim_{n \to \infty} \sqrt{\frac{3n^2 + 2n - 2}{2n^2 + 3n + 5}} = \sqrt{3/2}$$

$$\lim_{n \to \infty} \left(\frac{3n^2 + 2n - 2}{2n^2 + 3n + 5}\right)^2 = 9/4$$

$$\lim_{n \to \infty} \ln \left|\frac{3n^2 + 2n - 2}{2n^2 + 3n + 5}\right| = \ln(3/2).$$

(It might also be worth noting that the original a_n is given by $f(\frac{1}{n})$ where $f(x) = \frac{3+2x-2x^2}{2+3x+5x^2}$, which is continuous at $x = 0$.)

For our other application of calculus to sequences, we first remark that often the sequences whose limits we are trying to evaluate can be written in the form

$$a_n = f(n)$$

for some familiar function $f(x)$ which is actually defined for *all real numbers*, $x \in \mathbb{R}$. In such a case it can be useful to study

$$\lim_{x \to \infty} f(x)$$

to get information about $\lim_{n \to \infty} a_n$.

Indeed, recall from calculus that we have

Definition 1.5.12

Let $f(x)$ be a function defined for all real $x \in \mathbb{R}$ (or at least for all x larger than some real number a). Then we say $f(x)$ converges to the real number L, as x goes to infinity, if the following condition is true:

For every $\epsilon > 0$ there is an $R \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > R$.

In this case we write $\lim_{x \to \infty} f(x) = L$.

From this definition it is straightforward to prove (see exercise 9)

Proposition 1.5.13

Let f be defined on $\mathbb{R}$ and assume $\lim_{x \to \infty} f(x) = L$. Define the sequence $a_n = f(n)$, then $\lim_{n \to \infty} a_n = L$.

Well, the above observation doesn't really help much until we recall that there is a theorem from Calculus that helps us evaluate limits of the above type. First we need to discuss a special kind of divergence.

Definition 1.5.14

Let f be a function defined for all real $x \in \mathbb{R}$. Then we say $f(x)$ *diverges to infinity*, as x goes to infinity, if the following condition is true:

For every $M > 0$ there is an $R \in \mathbb{R}$ so that $f(x) > M$ whenever $x > R$.

In this case we write $\lim_{x \to \infty} f(x) = \infty$.

Example 1.5.15

The function $f(x) = x$ diverges to infinity as x goes to infinity, but the function $g(x) = x \sin(x)$ does not.

We have a similar definition of divergence to infinity for sequences.

Definition 1.5.16

The sequence $\{a_n\}$ *diverges to infinity* if for every $M > 0$ there is an $N \in \mathbb{N}$ so that $a_n > M$ for all $n > N$.

Now we can state L'Hôpital's rule. The proof can be found in many calculus textbooks.

L' Hôpital's Rule:

Suppose f and g are differentiable functions satisfying

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty \quad or \quad \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0.$$

Then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

as long as the limit on the right hand side exists (or diverges to infinity).

Example 1.5.17

Let $a_n = \ln(n)/n$ for all n. Then we can write $a_n = f(n)/g(n)$ where $f(x) = \ln(x)$ and $g(x) = x$ for all $x > 0$. Now $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to \infty} g(x) = \infty$ so we can apply L'Hôpital's Rule to get.

$$\lim_{x \to \infty} \frac{\ln(x)}{x} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

Finally, applying Proposition 1.5.13 we conclude that

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = 0.$$

Remark:

L'Hôpital's rule applies only to functions defined on $\mathbb{R}$ so we must first convert our questions about sequences into questions about functions on $\mathbb{R}$ before we apply this theorem. *It makes no sense to differentiate an expression like* $\ln(n)$, *so L'Hôpital's rule doesn't even make sense here.*

EXERCISES 1.5

1. For each of the following limits either evaluate or explain why it is divergent.

   a.) $\lim \frac{\ln(n)}{n}$ b.) $\lim \frac{e^n}{n}$

   c.) $\lim \frac{n}{e^n}$ d.) $\lim \frac{n^2}{e^n}$

   e.) $\lim \frac{n}{2^n}$ f.) $\lim \frac{n}{\ln(n)}$

   g.) $\lim \frac{\ln(n)}{n^2}$ h.) $\lim \frac{n^2+1}{n\ln(n)}$

   i.) $\lim \frac{\cos(n)}{n}$ j.) $\lim \frac{n}{\sin(n)}$

   k.) $\lim(n\sin(1/n))$ l.) $\lim(1 + \frac{1}{n})^n$

   m.) $\lim (1+n)^{\frac{1}{n}}$ n.) $\lim \sqrt{n+1} - \sqrt{n}$

   o.) $\lim \sqrt{n^2+n} - \sqrt{n^2-n}$

2. Fill in the details of part a.) of Proposition 1.5.3, i.e. assume that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ and prove that $\lim_{n\to\infty}(a_n + b_n) = L + M$.

3. Prove that if the sequence $\{a_n\}$ converges and the sequence $\{b_n\}$ diverges, then the sequence $\{a_n + b_n\}$ diverges. (Hint: $b_n = (a_n + b_n) - a_n$.)

4. Use the properties of limits to explain why $\lim \frac{3n+1}{2n+5} = 3/2$. State clearly where you are using each of the properties.

5. Evaluate the following limits.

   a.) $\lim \frac{3n^2-2n+7}{6n^2+3n+1}$

   b.) $\lim \left(\frac{3n^2-2n+7}{6n^2+3n+1}\right)^3$

   c.) $\lim \sqrt{\frac{3n^2-2n+7}{6n^2+3n+1}}$

   d.) $\lim \left(\exp\left(\frac{3n^2-2n+7}{6n^2+3n+1}\right)\right)$

6. a.) Prove that if $\lim_{n\to\infty} a_n = \infty$ then $\lim_{n\to\infty} 1/a_n = 0$.

   b.) Assume that $a_n \neq 0$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} a_n = 0$, then prove that $\lim_{n\to\infty} |1/a_n| = \infty$.

   c.) Provide an example of a sequence $\{a_n\}$ of nonzero real numbers for which $\lim_{n\to\infty} a_n = 0$ but $\lim_{n\to\infty} 1/a_n$ is neither ∞ nor $-\infty$.

7. In this exercise we prove the *Squeeze Lemma: Assume that $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are sequences with $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Assume also that $\lim a_n = \lim c_n = L$. Then $\lim b_n = L$.*
   We break the proof into two pieces:

   a.) Assume $\{r_n\}$ and $\{s_n\}$ are sequences with $0 \leq r_n \leq s_n$ and $\lim s_n = 0$. Use the definition of the limit to show that $\lim r_n = 0$.

   b.) Now consider sequences $\{a_n\}, \{b_n\}$, and $\{c_n\}$ with $a_n \leq b_n \leq c_n$ and $\lim a_n = \lim c_n = L$. Let $r_n = b_n - a_n$ and $s_n = c_n - a_n$ and show that part a.) can be applied to prove that $\lim r_n = 0$. Now use Proposition 1.5.3 to prove that $\lim b_n = L$.

8. Provide the details of the proof of Lemma 1.5.4.

9. Provide a proof for Proposition 1.5.13.

10. Prove that if $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{n\to\infty} a_n = \infty$.

11. Give the details of the proof for part c.) of Proposition 1.5.3 in the case that $M < 0$ and $a_n = 1$ for all n. (Careful, you will need to state and prove a new version of Lemma 1.5.4.)

12. Assume that f is a function defined on an open interval A containing $c \in \mathbb{R}$ and that it has the property that whenever there is a sequence

$\{a_n\}$ in A satisfying $\lim_{n \to \infty} a_n = c$ it happens that $\lim_{n \to \infty} f(a_n) = f(c)$. Prove that f is continuous at c (see definition 1.5.9).

1.6 Abstract Theorems on Convergence of Sequences: What is Reality?

So far, whenever we have shown that a sequence converges we have also determined the limit. In fact, it is not possible to use the definition of convergence without stating explicitly what the limiting value is. On the other hand, it seems reasonable that there might be criteria for convergence that don't require knowledge of the limiting value. A very intuitive example of this is the idea that if a sequence is decreasing and bounded below then it should converge to some number which is no less than the lower bound for the sequence.

Example 1.6.1 (see exercise 1.3.20 and example 1.4.1)

Let a_n be the sequence given recursively by $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$ with $a_1 = 2$. Then we can prove that this sequence is decreasing and that $a_n > 0$ for all $n \in \mathbb{N}$. Our intuition says that there must be some number L so that $\lim_{n \to \infty} a_n = L$. If this is true, then we would have $\lim_{n \to \infty} a_{n+1} = L$ and $\lim_{n \to \infty} \left(\frac{a_n}{2} + \frac{1}{a_n} \right) = \frac{L}{2} + \frac{1}{L}$ as well. But since $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$ we can conclude that $L = \frac{L}{2} + \frac{1}{L}$. Simplifying this yields $L^2 = 2$, so $L = \sqrt{2}$.

Although it seems intuitively clear that it should be the case that

$$\lim a_n = \sqrt{2},$$

we can't really prove this without having some understanding about what exactly we mean by the number $\sqrt{2}$.

The rational numbers $\mathbb{Q}$ are fairly easy to comprehend. We begin with the natural numbers (which are after all quite natural), and then tack on 0 and the negative integers (ok, these may be not quite so natural – for instance the Romans didn't even like to think of 0 as a number). Next, to get the rationals, either we think of dividing our natural numbers into equal portions or we think of comparing whole numbers by their ratios. The ancient Greeks

were happy with this concept, but they also knew that there are numbers that cannot be represented as a ratio of whole numbers. For example, although a segment of length $\sqrt{2}$ is easily constructed as the hypotenuse of a right triangle with legs of length 1, the Greeks had proven the following:

Proposition 1.6.2

There are no natural numbers a and b so that $\sqrt{2} = a/b$, i.e. $\sqrt{2}$ is not rational.

Proof: Let's suppose on the contrary that we have natural numbers a and b with $\sqrt{2} = a/b$. Of course, we can assume that a and b have no common divisors, since if they did we could "cancel" those divisors to get an equivalent fraction. That is, we assume that the fraction a/b is in "lowest terms". Now, if $a/b = \sqrt{2}$, then we have $a = \sqrt{2}b$, and so $a^2 = 2b^2$. Since $2b^2$ is an even number, we conclude that a^2 is even. But, the square of an odd number is odd, so if a^2 is even it must be the case that a is even too. Thus a can be written as $a = 2l$ for some natural number l. Squaring this we see that $a^2 = 4l^2$. Now we substitute back into the equation $a^2 = 2b^2$ to get $4l^2 = 2b^2$. Cancelling a factor of 2 yields $b^2 = 2l^2$. Now we can repeat the above argument to conclude that b is even too (since b^2 is even). Thus we have shown that both a and b are even, contradicting the assumption that a/b is in "lowest terms". Since this argument leads to a contradiction, we must conclude that the original assumption that $\sqrt{2} = a/b$ is incorrect.

$\square$

So, then, if $\sqrt{2}$ is not rational, what sort of number is it? If we punch $\sqrt{2}$ into our calculator we get a numerical readout like 1.414213562. But this is a rational number (represented in fraction form by $\frac{1,414,213,562}{1,000,000,000}$) so it can't really be $\sqrt{2}$! In fact, any finite decimal represents a rational number, so irrational numbers cannot have a finite decimal representation. Well then, as we have been taught in high school, the decimal representation for irrationals like $\sqrt{2}$ must "go on for ever". What is the meaning of decimals that "go on for ever"? One reasonable interpretation is that the decimal represents a sequence of better and better approximations of the actual number $\sqrt{2}$. If we define a sequence with

$$a_1 = 1, a_2 = 1.4, a_3 = 1.41, a_4 = 1.414, \; etc.$$

then we might say that $\sqrt{2}$ is *defined* to be the limit of this sequence. But why does this sequence have a limit if the thing it limits to is defined by the

limit? This all seems pretty circular but, in fact, if one is careful one can use these ideas to give a precise definition of the real numbers.

We will leave it for an appendix (to be written at some unknown future date) to study the precise definition of the real numbers, but we will discuss the idea of how this is done. The approach is to come up with a list of properties that we expect the real numbers to have, describe a set which has these properties, and then prove that the properties uniquely define the set up to a change of name. (It doesn't matter what symbols we use for our numbers, as long as we can provide a dictionary from one set of symbols to the other.) The list of properties include the many properties of numbers which we are familiar with from grade school. There are the algebraic properties such as commutativity, associativity and distributivity as well as the order properties i.e. the properties of inequalities (some are mentioned in section 1.3). Of course when you start listing properties of numbers you will notice that there are infinitely many things that you could write down but that many of the properties can actually be proven from some of the other properties – so the game is to try to find a finite list of properties from which all of the other properties follow. (This will be done in appendix A if it is ever written). However, after writing the list of algebraic and order properties of the real numbers it becomes clear that there must be some further property that distinguishes the real numbers from other sets of numbers, since, for instance, the rational numbers have all of the same algebraic and order properties as the real numbers. The missing ingredient is called the *Property of Completeness* which we will couch in the following terms:

Property of Completeness:
  Every bounded monotone sequence of real numbers is convergent.

Notice that the rational numbers do not enjoy this property. The two sequences above which limit to $\sqrt{2}$ are monotonic and bounded sequences of rational numbers, yet they do not limit to a rational number. As mentioned, we will leave it to an appendix to discuss the details of constructing the real numbers so that they satisfy this property. In this section we will simply accept the above property as a given fact and discuss some of its consequences. Our goal will be to come up with a *characterization* of convergent sequences which does not refer to the actual value of the limit. The above property gives a *criterion* for convergence without referring to the value of the limit, but it is not a characterization since there are many convergent sequences

that do not satisfy this criterion. (We know convergent sequences must be bounded, but we know of many non-monotonic sequences that converge.)

Example 1.6.3

Let's reconsider the sequence studied in example 1.6.1, given recursively by $a_1 = 2$, and $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$. Notice that if $a_n > 0$ then from the recursion formula it follows that $a_{n+1} > 0$, thus a straightforward induction argument proves that this sequence is bounded below by 0. We also claim that this sequence is decreasing. To see this we first need to note that $a_n^2 = \frac{a_{n-1}^2}{4} + \frac{1}{a_{n-1}^2} + 1$, so $a_n^2 - 2 = \frac{a_{n-1}^2}{4} + \frac{1}{a_{n-1}^2} - 1 = \left(\frac{a_{n-1}}{2} - \frac{1}{a_{n-1}}\right)^2 \geq 0$ for any $n \geq 2$. Thus we have that $a_n - a_{n+1} = \frac{a_n}{2} - \frac{1}{a_n} = \frac{a_n^2 - 2}{2a_n} \geq 0$ for all $n \geq 1$. That is, the sequence is decreasing. Now since we have verified that this sequence is decreasing and bounded below, we can conclude by the Property of Completeness that it must converge.

Our first result involves sequences of intervals in $\mathbb{R}$.

Proposition 1.6.4

Let $\{A_n\}$ be a sequence of closed, bounded, intervals in $\mathbb{R}$ with the property that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$. Then there is at least one real number α such that $\alpha \in A_n$ for every $n \in \mathbb{N}$.

Since A_n is a closed interval, we can write it in the interval notation $A_n = [a_n, b_n]$ where a_n is the left endpoint of the interval, and b_n is the right endpoint. In particular, we have the inequality

$$a_n \leq b_n.$$

Also, since we have $A_{n+1} \subset A_n$ we get the inequalities

$$a_n \leq a_{n+1}$$

and

$$b_{n+1} \leq b_n.$$

Now, the second of these three inequalities tells us that the sequence, $\{a_n\}$, of left endpoints, is increasing and the third inequality tells us that the sequence, $\{b_n\}$, of right endpoints is decreasing. In particular, since $\{b_n\}$ is decreasing we have that $b_n \leq b_1$ for all $n \in \mathbb{N}$, and so, using the first inequality, we conclude that

$$a_n \leq b_1 \quad \text{for all} \ n \in \mathbb{N}.$$

Now increasing sequences are always bounded below, so the sequence $\{a_n\}$ is monotone and bounded. Thus we can apply the Property of Completeness to conclude that the sequence $\{a_n\}$ converges to some real number, which we call α, i.e. $\lim a_n = \alpha$. To complete the proof we will show that $\alpha \in A_n$ for every $n \in \mathbb{N}$. This means that we need to show that $a_n \leq \alpha$ and $\alpha \leq b_n$ for every $n \in \mathbb{N}$. The first of these inequalities is a direct application of Proposition 1.4.16 and the second follows from Proposition 1.4.17 and the observation that for any fixed $m \in \mathbb{N}$ we have that $a_n \leq b_n \leq b_m$ for all $n \geq m$ and that $a_n \leq a_m \leq b_m$ for all $n \leq m$. Thus each b_m gives an upper bound for $\{a_n\}$.

$\square$

As a corollary to Proposition 1.6.4 we can prove

Proposition 1.6.5
    Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof: We begin the proof by using our bounded sequence to define a sequence of closed intervals, $\{A_n\}$, with $A_{n+1} \subset A_n$ for each $n \in \mathbb{N}$. First, we start with a sequence $\{a_n\}$ which is bounded, i.e., there is some real number M with the property that $|a_n| < M$ for all $n \in \mathbb{N}$. Let A_1 be the closed interval given by $A_1 = [-M, M]$. Now divide this interval in half to define the two subintervals B_1 and C_1 given by $B_1 = [-M, 0]$ and $C_1 = [0, M]$. Since all of the points in the sequence are contained in the interval A_1 and also $A_1 = B_1 \cup C_1$, it follows that at least one of the two intervals, B_1 or C_1 contains infinitely many of the a_n's. Define A_2 to be one of these subintervals which contains infinitely many of the a_n's. Notice that $A_2 \subset A_1$. Now do the same to A_2 as we did to A_1. Namely, divide it into two equal closed subintervals, B_2 and C_2. At least one of these two subintervals contains infinitely many of the a_n's, so we can choose A_3 to be one of these subintervals which does contain infinitely many of the a_n's. Again, we have $A_3 \subset A_2$. We can continue in this fashion for ever; since we always choose A_i to contain infinitely many of the a_n's we are guaranteed that the process does not stop. Thus we have a sequence of closed intervals satisfying the hypothesis of Proposition 1.6.4, guaranteeing the result that there is at least one real number which is contained in all of the A_n's. Choose such a number and call it α. Also notice that the length of A_1 is $2M$, the length of A_2 is one half of the length of A_1, i.e. M, etc. In general, the length of A_n is $2^{2-n}M$.

The next step of the proof is to describe how we pick an appropriate subsequence $\{b_n\}$ of $\{a_n\}$. Recall that a subsequence is determined by a

strictly increasing function $g : \mathbb{N} \to \mathbb{N}$, so to describe $\{b_n\}$ we need only describe g. To begin with, let $g(1) = 1$ (so $b_1 = a_1$). Next, to define $g(2)$, choose any a_{k_2} in the interval A_2 with $k_2 > 1$. Let $g(2) = k_2$ (so $b_2 = a_{k_2}$). To define $g(3)$, choose any $a_{k_3} \in A_3$ with $k_3 > g(2)$ (we know there are such since A_3 contains infinitely many of the a_n's). Let $g(3) = k_3$. Continue this process; once we have chosen $g(n)$ we can select any $a_{k_{n+1}} \in A_{n+1}$ with $k_{n+1} > g(n)$, let $g(n+1) = k_{n+1}$ (so, in general, $b_n = a_{k_n}$).

The final step of this proof is to show that this subsequence $\{b_n\}$ is indeed convergent. In fact, we will show that this subsequence converges to the number α which we chose in the first part of this proof. Let $\epsilon > 0$ be given. Since we know that the sequence $\{M2^{2-n}\}$ converges to zero, we can chose an $N \in \mathbb{N}$ so that $M2^{2-N} < \epsilon$. But if $n > N$ we know that $b_n \in A_n \subset A_N$ and of course that also $\alpha \in A_N$ as well. So we see that $|b_n - \alpha|$ is less than the length of A_N, which in turn is less than ϵ, i.e. $|b_n - \alpha| < \epsilon$ for all $n > N$.
□

Remark:

Since α was chosen to be any point which is contained in all of the intervals A_n, but turns out to be the limit of this subsequence (which we know to be unique), we can conclude that for this sequence of closed intervals, there is exactly one α which is contained in all of the A_n's.

Example 1.6.6

The sequence given by $a_n = \sin(n)$ is bounded between -1 and 1 but is certainly not convergent. As a challenge try to find an explicit subsequence which does converge.

Now we state the characterization of convergence to which we alluded above.

Definition 1.6.7

The sequence $\{a_n\}$ is called a *Cauchy sequence* if it satisfies the following: for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $|a_n - a_m| < \epsilon$ for all $n, m > N$.

Proposition 1.6.8

Every convergent sequence is a Cauchy sequence.

Proof: We leave the proof of this statement to the exercises, but here is a hint: assume $\lim a_n = L$ and use the triangle inequality to get $|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L|$.
□

Proposition 1.6.9

Every Cauchy sequence is bounded.

Proof: Again we leave the proof for the exercises. The hint here is simply to take $\epsilon = 1$ and emulate the proof of Proposition 1.4.10.

<div align="right">□</div>

Proposition 1.6.10

Every Cauchy sequence of real numbers converges to some real number.

Proof: Let $\{a_n\}$ be a Cauchy sequence. Since we have proven that every Cauchy sequence is bounded (proposition 1.6.9) and that every bounded sequence of real numbers has a convergent subsequence (proposition 1.6.5) we can conclude that every Cauchy sequence of real numbers has a convergent subsequence. Thus we know that $\{a_n\}$ has a convergent subsequence, $\{b_n\}$.

Say $\lim b_n = L$. We will show that $\lim a_n = L$ also. First let $\epsilon > 0$ be given and, using the Cauchy property, choose $N \in \mathbb{N}$ so large that $|a_n - a_m| < \epsilon/2$ whenever n and m are both greater than N. Now since the b_n's are a subsequence of the a_n's we know that for each k there is an $n_k \geq k$ so that $b_k = a_{n_k}$. From this it follows that whenever n and m are both greater than N we have $|a_n - b_m| < \epsilon/2$. We also know that $\lim b_n = L$, so we can find some $\tilde{N} \in \mathbb{N}$ so large that $|b_n - L| < \epsilon/2$ whenever $n > \tilde{N}$. Now fix M to be the maximum of $N + 1$ and $\tilde{N} + 1$, then we have, when $n > M$,

$$\begin{aligned}
|a_n - L| &= |a_n - b_M + b_M - L| \\
&\leq |a_n - b_M| + |b_M - L| \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

<div align="right">□</div>

<div align="center">

EXERCISES 1.6

</div>

1. a.) Suppose that a is a natural number and that a^2 is divisible by 3, show that a is divisible by 3. (Hint: show that if a is not divisible by 3, then neither is a^2.)

    b.) Show that $\sqrt{3}$ is not rational.

2. Prove that every Cauchy sequence is bounded.

3. Prove that every convergent sequence is a Cauchy sequence.

4. Use Newton's method to construct a sequence that converges to $\sqrt{3}$. Prove that your sequence is convergent to $\sqrt{3}$. (See examples 1.4.1 and 1.6.3.)

5. Let $\{a_n\}$ be the sequence defined by $a_1 = 1$ and $a_{n+1} = a_n + \frac{1}{(n+1)^2}$. Prove that this sequence converges. (Hint: use induction to prove that $a_n \leq 2 - 1/n$.)

6. For each $n \in \mathbb{N}$ let A_n be the interval $A_n = [n, \infty)$. Show that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ and yet there is no number $\alpha \in \mathbb{R}$ with $\alpha \in A_n$ for all n. Why does this not contradict proposition 1.6.4?

7. For each $n \in \mathbb{N}$ let A_n be the interval $A_n = (0, 1/n]$. Show that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ and yet there is no number $\alpha \in \mathbb{R}$ with $\alpha \in A_n$ for all n. Why does this not contradict proposition 1.6.4?

8. Let $\{a_n\}$ be a sequence which is bounded above and define a new sequence $\{b_n\}$ by $b_n = \max\{a_1, a_2, \ldots, a_n\}$. (The *maximum* of a finite set A is the element of A which is greater than or equal to all elements of A.) Warning: the sequence $\{b_n\}$ may not be a subsequence of $\{a_n\}$.

   a.) Show that the sequence $\{b_n\}$ is increasing. (*Hint:* Notice that if if A and B are finite sets of real numbers with $A \subseteq B$, then $\max A \leq \max B$.)

   b.) Assume that U is an upper bound for $\{a_n\}$, show that U is an upper bound for $\{b_n\}$.

   c.) From the Property of Completeness we know that $\{b_n\}$ must have a limit, call it L. Prove that $a_n \leq L$ for all $n \in \mathbb{N}$, i.e. L is an upper bound for $\{a_n\}$. Also prove that if U is any upper bound for $\{a_n\}$ then $L \leq U$. Thus L is the *least upper bound* of the sequence $\{a_n\}$. We denote this least upper bound by $\sup a_n$, where sup is short for the Latin word *supremum* .

   Similarly, if the sequence $\{a_n\}$ is bounded below, we can use the Property of Completeness to prove that it has a *greatest lower bound* . We denote this by $\inf a_n$, where inf is short for the Latin word *infimum* .

9. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence. For each $k \in \mathbb{N}$ consider the subsequence which begins with a_k and continues on from there, i.e. $\{a_n\}_{n=k}^{\infty}$.

Let $L_k = \sup_{n \geq k} a_n$ denote the least upper bound of the kth such subsequence. (This least upper bound is shown to exist in exercise 8.)

a.) Prove that the sequence $\{L_k\}$ is decreasing.

b.) Prove that if M is a lower bound for the sequence $\{a_n\}$ then it is also a lower bound for the sequence $\{L_k\}$.

From the Property of Completeness, we conclude that the sequence $\{L_k\}$ has a limit, $\inf L_k$. This limit is called the *limsup* of the sequence $\{a_n\}$, and is denoted by $\limsup a_n$. Notice that

$$\limsup a_n = \inf_{k \geq 1} \sup_{n \geq k} a_n.$$

10. Let $\{a_n\}$ be a bounded sequence and let $L = \limsup a_n$.

a.) Show that there is a convergent subsequence, $\{b_n\}$, of $\{a_n\}$ with $\lim b_n = L$. (Note: this gives an alternative proof that every bounded sequence has a convergent subsequence.)

b.) Prove that if $\{c_n\}$ is any convergent subsequence of $\{a_n\}$ then $\lim c_n \leq L$.

11. Let $S \subset \mathbb{R}$ be a nonempty set which is bounded above. The real number α is called the *least upper bound of S* if it satisfies the following two properties:

i.) If $s \in S$, then $s \leq \alpha$ (so α is an upper bound for S).

ii.) If U is an upper bound for S, then $\alpha \leq U$.

In this exercise we will prove that every nonempty subset of $\mathbb{R}$ which is bounded above has a least upper bound. This is called the *Least Upper Bound Property* of the real numbers.

a.) Let $S \subset \mathbb{R}$ be a nonempty subset which is bounded above and let $\epsilon > 0$ be given. Prove that there is an element $a \in S$ so that $a + \epsilon$ is an upper bound for S. (Hint: Pick $a_0 \in S$ and find $m \in \mathbb{N}$ so that $a_0 + m\epsilon$ is an upper bound for S but $a_0 + (m-1)\epsilon$ is not an upper bound for S. Show that you can find an $a \in S$ such that $a_0 + (m-1)\epsilon \leq a \leq a_0 + m\epsilon$.)

b.) Use part a.) to construct an increasing sequence $\{a_n\}$ in S with the property that $a_n + 1/n$ is an upper bound for S.

c.) Explain why the sequence $\{a_n\}$ is bounded above and thus has a limit, $\lim_{n\to\infty} a_n = \alpha$.

d.) Prove that the number α constructed above is a least upper bound for S by showing that α satisfies properties i.) and ii.) above.

12. Let $\{a_n\}$ be an increasing sequence which is bounded above. Using the Least Upper Bound property from 11, let α be the least upper bound of the set of values $\{a_n | n \in \mathbb{N}\}$. Without using the Property of Completeness, show that $\lim a_n = \alpha$.

Remark: In exercise 11 we used the Property of Completeness to prove the Least Upper Bound property of the real numbers. In this exercise we have turned the tables; we assumed that the real numbers satisfy the Least Upper Bound property and showed that the Property of Completeness follows. This shows that these two properties are logically equivalent; if we know one of the properties is true, then so must be the other.

13. Recall that the Archimedean Property says that the natural numbers are not bounded. Use the Least Upper Bound property of the real numbers to prove this fact. (*Hint:* Proceed by contradiction.)

1.7 Some Results from Calculus

Armed with a deeper knowledge of the real numbers, we are now able to prove some of the most important results about continuous functions on $\mathbb{R}$ that are usually stated without proof in introductory calculus courses. Before stating the first of these, we need to define continuity on an interval. (The definition of continuity at a point is given in 1.5.9.)

Definition 1.7.1
A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous on the open interval, (a, b),* if it is continuous at every point of (a, b). The function f is *continuous on the closed interval, $[a, b]$* if it is continuous on the open interval (a, b) and *continuous from the right* at a and *continuous from the left* at b.

Of course to make sense of the above, we need the following:

Definition 1.7.2
A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous from the right* at a if for every $\epsilon > 0$ there is a $\delta > 0$ so that $|f(x) - f(a)| < \epsilon$ whenever $0 \leq x - a < \delta$. (Thus x is within δ of a, but to the right of a.)

Similarly, f is said to be *continuous from the left* at b if for every $\epsilon > 0$ there is a $\delta > 0$ so that $|f(x) - f(b)| < \epsilon$ whenever $0 \leq b - x < \delta$.

The first theorem of this chapter is known as the *Intermediate Value Theorem*. This theorem is often prosaically interpreted by saying that you can always draw the graph of a continuous function on a closed interval without lifting your pencil from the paper. A deeper interpretation is that this theorem is intimately tied to the fact that there are "no holes" in the real number line, i.e. the completeness axiom.

Theorem 1.7.3
Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous on the closed interval $[a, b]$, and that $f(a) < 0$ and $f(b) > 0$. Then there is some point $c \in (a, b)$ such that $f(c) = 0$.

Remark: In the exercises we will outline how to generalize this result to show that, with no restrictions on $f(a)$ and $f(b)$, the function f must attain every value between the numbers $f(a)$ and $f(b)$.

Proof: The proof is very similar to that of Proposition 1.6.4 in that we will successively subdivide the interval to define a sequence of numbers, $\{a_n\}$, in $[a, b]$ which must converge. To do this, begin by naming the interval $[a, b]$ to be J_1 and let $a_1 = a$, and $m_1 = (b + a)/2$, the midpoint of the interval J_1. There are three cases to consider:

1.) If $f(m_1) = 0$ we can stop, by taking $c = m_1$.

2.) If $f(m_1) > 0$ define J_2 to be the interval $[a, m_1]$ and define $a_2 = a$ and $b_2 = m_1$.

3.) If $f(m_1) < 0$ we take J_2 to be the interval $[m_1, b]$, and define $a_2 = m_1$ and $b_2 = b$.

Notice that $J_2 = [a_2, b_2]$ and $f(a_2) < 0$, and $f(b_2) > 0$. Thus f satisfies the conditions of the theorem on J_2. Also notice that $b_2 - a_2 = (b_1 - a_1)/2$, i.e. the length of J_2 is half of the length of J_1.

Proceeding inductively, suppose that we have found an interval $J_k = [a_k, b_k]$ with $f(a_k) < 0$, $f(b_k) > 0$, and the length of J_k equal to $1/2^{k-1}$ times

the length of J_1. Let $m_k = (b_k + a_k)/2$ and consider $f(m_k)$. Again, we consider three cases:

    1.) If $f(m_k) = 0$, we can stop, and let $c = m_k$.

    2.) If $f(m_k) > 0$ define J_{k+1} to be the interval $[a, m_k]$ and define $a_{k+1} = a_k$ and $b_{k+1} = m_k$.

    3.) If $f(m_k) < 0$ we take J_{k+1} to be the interval $[m_k, b]$, and define $a_{k+1} = m_k$ and $b_{k+1} = b_k$.

By this procedure, we either stop at some step, having already found a point at which f vanishes, or we continue forever, producing an infinite sequence of nested intervals, $\{J_k\}$, with $J_k = [a_k, b_k]$ and $f(a_k) < 0$, $f(b_k) > 0$, and $b_{k+1} - a_{k+1} = (b_k - a_k)/2^k$. It is now straightforward to check that the sequence of left endpoints, $\{a_k\}$, is increasing and bounded above, while the sequence of right endpoints, $\{b_k\}$ is decreasing and bounded below. Thus, by the completeness axiom, these two sequences have limits. Notice also that $\lim(b_k - a_k) = 0$, so the two sequences limit to the same point. Call the limit point c. It remains to show that $f(c) = 0$. Since f is continuous, we know, by proposition 1.5.10 that

$$f(c) = f(\lim a_k) = \lim f(a_k).$$

But $f(a_k) < 0$ for every k, so $\lim f(a_k) \leq 0$. Likewise

$$f(c) = f(\lim b_k) = \lim f(b_k).$$

And now $f(b_k) > 0$ for every k, so $\lim f(b_k) \geq 0$. Having thus shown that $f(c) \leq 0$ and $f(c) \geq 0$, we conclude that $f(c) = 0$.

                                                            □

Our next theorem states that every continuous function f on a closed interval must be bounded above. (Of course, by applying the theorem to $-f$, one can conclude that it is also bounded below as well.)

Theorem 1.7.4

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then there exists an $M \in \mathbb{R}$ so that $f(x) \leq M$ for all $x \in [a, b]$.

Proof: We proceed by contradiction. Assume to the contrary that no such M exists. Then for every $n \in \mathbb{N}$, there is some point $a_n \in [a, b]$ so that $f(a_n) > n$. The sequence $\{a_n\}$ is bounded, so by Proposition 1.6.5 we conclude that there must be some subsequence, $\{b_k\}$, which converges to some number c. Since the b_k's are all in $[a, b]$, i.e. $a \leq b_k \leq b$ it follows that $a \leq c \leq b$, i.e. $c \in [a, b]$.

Since f is continuous, we know that $f(c) = \lim f(b_k)$, however, it is clear that $\lim_{k \to \infty} f(a_k) = \infty$ and so $\lim_{k \to \infty} f(b_k) = \infty$ as well. This is a contradiction.

$\square$

To the reader: Why is this a contradiction?

The final theorem of this section gives the theoretical foundation of the optimization problems from freshman calculus. In those problems you were given a function on a closed interval, $f : [a, b] \to \mathbb{R}$, and asked to find $c \in [a, b]$ so that $f(c)$ is the maximum value of f on $[a, b]$, i.e. $f(c) \geq f(x)$ for all $x \in [a, b]$. This theorem says that such a c must exist.

Theorem 1.7.5
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then there exists $c \in [a, b]$ such that $f(c) \geq f(x)$, for all $x \in [a, b]$.

Proof As usual, we will try to construct a sequence that will converge to a number c that satisfies the required conditions. First, we show that, given $\epsilon > 0$, we can find a number $\tilde{c}$ which satisfies the requirements "within ϵ." That is, we claim that there is a number $\tilde{c} \in [a, b]$ so that $f(\tilde{c}) + \epsilon \geq f(x)$, for all $x \in [a, b]$. To prove this, consider the sequence $b_m = f(a) + m\epsilon$ for $m = 0, 1, 2, \ldots$. Now if $f(a) \geq f(x)$ for all $x \in [a, b]$ we can take $c = a$ and be done with our proof. If this is not the case, then we notice that the sequence $\{b_m\}$ is increasing and not bounded above. However, since the values of f are bounded above (by Theorem 1.7.4), there most be some value $m_0 \in \mathbb{N}$ so that $b_{m_0} \geq f(x)$ for all $x \in [a, b]$ but that b_{m_0-1} does not satisfy this, i.e. there is at least one $x \in [a, b]$ so that $f(x) > b_{m_0-1}$. Choose such an x and call it $\tilde{c}$. Then we have $f(\tilde{c}) + \epsilon > b_{m_0-1} + \epsilon = b_{m_0}$. And since $b_{m_0} \geq f(x)$ for all $x \in [a, b]$, we conclude that $f(\tilde{c}) + \epsilon \geq f(x)$, for all $x \in [a, b]$.

Now, for each $n \in \mathbb{N}$ we can use the above argument to choose a number $r_n \in [a, b]$ so that $f(r_n) + \dfrac{1}{n} \geq f(x)$, for all $x \in [a, b]$. This sequence is bounded, so by Proposition 1.6.5 it must have a convergent subsequence, let's call the subsequence $\{c_n\}$ and let the limit of this sequence be called c. Notice that since the sequence $\{c_n\}$ is a subsequence of $\{r_n\}$ we have that $c_n + \frac{1}{n} = r_m + \frac{1}{n}$ for some $m \geq n$, and so $f(c_n) + \frac{1}{n} \geq f(r_m) + \frac{1}{m} \geq f(x)$, for all $x \in [a, b]$. We also have that $c \in [a, b]$, since for each n, $c_n \in [a, b]$.

It remains to show that $f(c) \geq f(x)$ for all $x \in [a, b]$. Because we assume that the function f is continuous, we know that the sequence $\{f(c_n)\}$ limits to $f(c)$. Now let $\epsilon > 0$ be given and chose $N \in \mathbb{N}$ so that $|f(c_N) - f(c)| < \epsilon/2$

and so that $\frac{1}{N} < \epsilon/2$. Then we have

$$f(c) - \epsilon/2 < f(c_N) < f(c) + \epsilon/2.$$

Adding $\epsilon/2$ to the second inequality, we can conclude that

$$f(c) + \epsilon > f(c_N) + \epsilon/2 > f(c_N) + \frac{1}{N} \geq f(x),$$

for all $x \in [a, b]$.

Now for each fixed $x \in [a, b]$ we have shown that $f(c) + \epsilon \geq f(x)$. Since ϵ is an arbitrary positive number, we conclude that $f(c) \geq f(x)$. □

EXERCISES 1.7

1. Let $f(x)$ be a continuous function on the closed interval $[a, b]$ and let α be a real number between $f(a)$ and $f(b)$, i.e. $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$. Show that there is some value, $c \in (a, b)$ such that $f(c) = \alpha$. (Hint: if $f(a) < f(b)$ apply Theorem 1.7.3 to $f(x) - \alpha$ - make sure you check all of the hypotheses.)

2. Prove that there is a real number x so that $\sin(x) = x - 1$.

3. Use Theorem 1.7.3 to prove that every positive real number, $r > 0$, has a square root. (Hint: Consider $f(x) = x^2 - r$ on a suitable interval.)

4. Assume that f is a continuous function on $[0, 1]$ and that $f(x) \in [0, 1]$ for each x. Show that there is a $c \in [0, 1]$ so that $f(c) = c$.

5. Suppose that f and g are continuous on $[a, b]$ and that $f(a) > g(a)$ and $g(b) > f(b)$. Prove that there is some number $c \in (a, b)$ so that $f(c) = g(c)$.

6. a.) Give an example of a continuous function, f, which is defined on the open interval $(0, 1)$ which is not bounded above.

   b.) Give an example of a continuous function, f, which defined on the open interval $(0, 1)$ and is bounded above, but does not achieve its maximum on $(0, 1)$. I.e. there is no number $c \in (0, 1)$ satisfying $f(c) \geq f(x)$ for all $x \in (0, 1)$.

7. a.) Give an example of a function, f, which is defined on the closed interval $[0, 1]$ but is not bounded above (on $[0, 1]$).

b.) Give an example of a function, f, which is defined on the closed interval $[0, 1]$ and is bounded above, but does not have a maximal value on $[0, 1]$.

Chapter 2

Series

2.1 Introduction to Series

In common parlance the words series and sequence are essentially synonomous, however, in mathematics the distinction between the two is that a series is the sum of the terms of a sequence.

Definition 2.1.1

Let $\{a_n\}$ be a sequence and define a new sequence $\{s_n\}$ by the recursion relation $s_1 = a_1$, and $s_{n+1} = s_n + a_{n+1}$. The sequence $\{s_n\}$ is called the *sequence of partial sums of* $\{a_n\}$.

Another way to think about s_n is that it is given by the sum of the first n terms of the sequence $\{a_n\}$, namely

$$s_n = a_1 + a_2 + \ldots + a_n.$$

A shorthand form of writing this sum is by using the *sigma notation*:

$$s_n = \sum_{j=1}^{n} a_j.$$

This is read as s_n *equals the sum from* j *equals one to* n *of a sub* j. We use the subscript j on the terms a_j (instead of n) because this is denoting an arbitrary term in the sequence while n is being used to denote how far we sum the sequence.

Example 2.1.2

Using sigma notation, the sum $1 + 2 + 3 + 4 + 5$ can be written as $\sum_{j=1}^{5} j$.

It can also be denoted $\sum_{n=1}^{5} n$, or $\sum_{n=0}^{4} (n+1)$. Similarly, the sum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

can be written as $\sum_{j=2}^{6} \frac{1}{j}$, or $\sum_{n=2}^{6} \frac{1}{n}$, or $\sum_{n=1}^{5} \frac{1}{n+1}$. On the other hand, the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$$

can be written as $\sum_{j=1}^{n} \frac{1}{j}$ or $\sum_{k=1}^{n} \frac{1}{k}$ but *can not be written as* $\sum_{n=1}^{n} \frac{1}{n}$.

Definition 2.1.3

Let $\{a_n\}$ be a sequence and let $\{s_n\}$ be the sequence of partial sums of $\{a_n\}$. If $\{s_n\}$ converges we say that $\{a_n\}$ is *summable*. In this case, we denote the $\lim_{n\to\infty} s_n$ by

$$\sum_{j=1}^{\infty} a_j.$$

.

Definition 2.1.4

The expression $\sum_{j=1}^{\infty} a_j$ is called an *infinite series* (whether or not the sequence $\{a_n\}$ is summable). When we are given an infinite series $\sum_{j=1}^{\infty} a_j$ the sequence $\{a_n\}$ is called the *sequence of terms*. If the sequence of terms is summable, the infinite series is said to be *convergent*. If it is not convergent it is said to *diverge*.

Example 2.1.5

Consider the sequence of terms given by

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Then

$$s_1 = a_1 = 1 - \frac{1}{2} = \frac{1}{2},$$

$$s_2 = a_1 + a_2 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3})$$

$$= 1 + (\frac{1}{2} - \frac{1}{2}) - \frac{1}{3}$$

$$= \frac{2}{3},$$

$$s_3 = a_1 + a_2 + a_3 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4})$$

$$= 1 + (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{3}) - \frac{1}{4}$$

$$= \frac{3}{4},$$

etc. Continuing this regrouping, we see that

$$s_n = a_1 + a_2 + \cdots + a_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n+1})$$

$$= 1 + (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n}) - \frac{1}{n+1}$$

$$= \frac{n}{n+1}.$$

Therefore we see that the $\lim_{n \to \infty} s_n = 1$ and so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

Example 2.1.6
  Let $a_n = \frac{1}{2^n}$. Then

$$s_1 = a_1 = \frac{1}{2}$$

$$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_3 = a_1 + a_2 + a_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8},$$

etc. A straightforward induction argument shows that, in general,

$$s_n = 1 - \frac{1}{2^n}.$$

Thus $\lim s_n = 1$, and so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Example 2.1.7

Let $a_n = \frac{1}{n^2}$. Then the partial sum s_n is given by

$$s_n = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2}.$$

In particular, $s_1 = 1$, $s_2 = 5/4$, and $s_3 = 49/36$. Since the a_n's are all positive the s_n's form an increasing sequence. (Indeed, $s_n - s_{n-1} = a_n = \frac{1}{n^2} > 0$.) It is not much harder to show by induction (see exercise 7) that for each n, $s_n < 2 - \frac{1}{n}$ so that the sequence $\{s_n\}$ is bounded above by 2. Thus we conclude (from the Property of Completeness) that the sequence $\{a_n\}$ is summable, i.e. the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Remark: It is important to note here that although we have proven that this series converges, we do not know the value of the sum.

Example 2.1.8

Let $a_n = \frac{1}{n}$. Then the partial sum s_n is given by

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

In particular, $s_1 = 1, s_2 = 3/2$, and $s_3 = 11/6$. Again, notice that the s_n's are increasing. However, we will show that in this case they are not bounded above, hence the *harmonic* series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. To see that the partial sums are not bounded above we focus our attention on the subsequence $\sigma_n = s_{g(n)}$ where $g(n) = 2^{n-1}$. So we have

$$\sigma_1 = s_1 = 1,$$
$$\sigma_2 = s_2 = \frac{3}{2},$$
$$\sigma_3 = s_4 = \frac{25}{12},$$

etc. We will show that $\sigma_n \geq \sigma_{n-1}+1/2$. From this it follows by induction that $\sigma_n > n/2$ and so the sequence $\{\sigma_n\}$ is unbounded. To see that $\sigma_n \geq \sigma_{n-1}+1/2$ notice that $\sigma_n - \sigma_{n-1}$ is given by adding up the a_j's up to $a_{2^{n-1}}$ and then subtracting all of them up to $a_{2^{n-2}}$ so the net result is that

$$\sigma_n - \sigma_{n-1} = a_{2^{n-2}+1} + a_{2^{n-2}+2} + \ldots + a_{2^{n-1}}.$$

Now there are exactly $2^{n-1} - 2^{n-2} = 2^{n-2}$ terms in the sum on the right hand side and each of them is larger than or equal to $a_{2^{n-1}} = \dfrac{1}{2^{n-1}}$. Thus the $\sigma_n - \sigma_{n-1}$ is larger than or equal to $\dfrac{2^{n-2}}{2^{n-1}} = \dfrac{1}{2}$.

The following picture might be helpful in understanding the above argument;

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots$$

$$> \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots.$$

Example 2.1.9

Let $\{b_n\}$ be a given sequence. We can build a sequence $\{a_n\}$ whose sequence of partial sums is given by $\{b_n\}$ in the following way: Let $a_1 = b_1$ and for $n > 1$ let $a_n = b_n - b_{n-1}$. Then we have $a_1 + a_2 = b_1 + (b_2 - b_1) = b_2$, $a_1 + a_2 + a_3 = b_1 + (b_2 - b_1) + (b_3 - b_2) = b_3$, etc. Thus the sequence $\{b_n\}$ converges if and only if the sequence $\{a_n\}$ is summable.

The above example shows that a sequence is convergent if and only if a related sequence is summable. Similarly, a sequence is summable if and only if the sequence of partial sums converges. However it should be kept in mind that the sequence given by $a_n = 1/n$ converges but is *not* summable. On the other hand we have

Proposition 2.1.10

If the sequence $\{a_n\}$ is summable then $\lim a_n = 0$.

Proof: Let $\{s_n\}$ denote the sequence of partial sums of $\{a_n\}$. Then we know that $\lim s_n$ exists, call it S. Notice that $a_n = s_n - s_{n-1}$ so by the algebra of limits

$$\lim a_n = \lim s_n - \lim s_{n-1} = S - S = 0.$$

$\square$

We should mention that there is another useful result along these lines. Assume that $\{a_n\}$ is summable and let s denote the sum of the series. If s_n denotes the nth partial sum then we know that s_n limits to s. In other words

$$\lim_{n\to\infty} (s - s_n) = 0.$$

Thinking of this in summation notation,

$$s - s_n = \sum_{j=n+1}^{\infty} a_j.$$

This latter sum is sometimes called the *tail* of the series. Thus, if $\sum a_n$ converges, the tails of the series must limit to zero. We state this as

Proposition 2.1.11
Let $\{a_n\}$ be a sequence of real numbers and define the *sequence of tails* by

$$t_n = \sum_{j=n+1}^{\infty} a_j.$$

Then $\{a_n\}$ is summable if and only if $\lim_{n\to\infty} t_n = 0$.

We should also mention that, as with sequences, convergent series behave well with respect to sums and multiplication by a fixed real number (scalar multiplication). Of course multiplication of two series is more complicated, since even for finite sums, the product of two sums is not simply the sum of the products. We will return to a discussion of products of series in section 2.3. For now we state the result for sums and scalar multiplication of series, leaving the proofs to exercises 2.1.12 and 2.1.13.

Proposition 2.1.12
Let $\{a_n\}$ and $\{b_n\}$ be summable sequences and let r be a real number. Define two new sequences by $c_n = a_n + b_n$ and $d_n = ra_n$. Then $\{c_n\}$ and $\{d_n\}$ are both summable and

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$
$$\sum_{n=1}^{\infty} d_n = r \sum_{n=1}^{\infty} a_n.$$

We conclude this section with an important example called the *geometric series*.

Definition 2.1.13

Let $r \in \mathbb{R}$. The series $\sum_{n=1}^{\infty} r^n$ is called a *geometric series*.

Proposition 2.1.14

The geometric series $\sum_{n=1}^{\infty} r^n$ converges if and only if $|r| < 1$. In the case that $|r| < 1$ we have

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

Proof: First note that the terms of this series are given by $a_n = r^n$ and this sequence converges to zero if and only if $|r| < 1$. Hence if $|r| \geq 1$ we conclude from proposition 2.1.10 that the series diverges.

Now if $|r| < 1$ consider the partial sum

$$s_n = r + r^2 + r^3 + \ldots + r^n.$$

Notice that

$$r s_n = r^2 + r^3 + r^4 \ldots + r^{n+1} = s_n - r + r^{n+1}.$$

Solving for s_n yields

$$s_n = \frac{r - r^{n+1}}{1-r} = \frac{r}{1-r} - \frac{r^{n+1}}{1-r}.$$

Now if $|r| < 1$ we know that $\lim r^n = 0$ and hence $\lim \frac{r^{n+1}}{1-r} = 0$. Thus $\lim s_n = \frac{r}{1-r}$.

$\square$

Example 2.1.15

a.) The sum $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges to $\frac{1/5}{1 - 1/5} = \frac{1/5}{4/5} = 1/4$.

b.) The sum $\sum_{n=1}^{\infty} \frac{3}{5^n}$ equals $3 \sum_{n=1}^{\infty} \frac{1}{5^n}$ and so converges to $3/4$.

c.) The sum $\sum_{n=3}^{\infty} \frac{1}{5^n}$ equals $(\sum_{n=1}^{\infty} \frac{1}{5^n}) - (\frac{1}{5} + \frac{1}{5^2})$ and so converges to $1/4 - 1/5 - 1/25 = 1/100$.

EXERCISES 2.1

1. Consider the sequence given by $a_n = \frac{1}{2^n}$. Compute the first five partial sums of this sequence.

2. Rewrite the following sums using sigma notation:

   a.) $(1 + 4 + 9 + 16 + 25 + 36 + 49)$

   b.) $(5 + 6 + 7 + 8 + 9 + 10)$

   c.) $(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + ... + \frac{1}{28})$

   d.) $(2 + 2 + 2 + 2 + 2 + 2 + 2 + 2)$

3. Evaluate the following finite sums:

   a.) $\sum_{k=1}^{n} 1$ b.) $\sum_{k=1}^{n} 1/n$ c.) $\sum_{k=1}^{n} k$

   d.) $\sum_{k=1}^{2n} k$ e.) $\sum_{k=1}^{n} k^2$

4. Consider the sequence whose n^{th} term is $a_n = \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2}, n \in \mathbb{N}$. Compute the first five partial sums of this sequence. What is the general formula for the nth partial sum? Prove this formula by induction and prove that the sequence is summable.

5. Consider the sequence whose n^{th} term is $a_n = \frac{1}{(n+2)} - \frac{1}{(n+3)}, n \in \mathbb{N}$. Compute the first five partial sums of this sequence. What is the general formula for the nth partial sum? Prove this formula by induction and prove that the sequence is summable.

6. Consider the sequence whose n^{th} term is $a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}, n \in \mathbb{N}$. Prove that this sequence is summable.

7. Let $s_n = \sum_{j=1}^{n} \frac{1}{j^2}$. Prove by induction that $s_n \le 2 - \frac{1}{n}$. (Hint: Prove that $-\frac{1}{k} + \frac{1}{(k+1)^2} = -\frac{1}{k+1} - \frac{1}{k(k+1)^2}$.) (See example 2.1.7.)

8. Let $s_n = \sum_{j=1}^{n} \frac{1}{j^3}$. Prove by induction that $s_n \leq 2 - \frac{1}{n^2}$. Conclude that

$$\sum_{j=1}^{\infty} \frac{1}{j^3} \text{ converges.}$$

9. Prove that the sequence given by $a_n = \frac{n}{n+1}$ is not summable.

10. Find a sequence $\{a_n\}$ whose n^{th} partial sum is $\frac{n-1}{n+1}$.

11. Find a sequence $\{a_n\}$ whose n^{th} partial sum is the nth term in the Fibonacci sequence.

12. Let $\{a_n\}$ be a summable sequence and r a real number. Define a new sequence $\{b_n\}$ by $b_n = r a_n$. Prove that $\{b_n\}$ is summable and that

$$\sum_{n=1}^{\infty} b_n = r \sum_{n=1}^{\infty} a_n.$$

13. Let $\{a_n\}$ and $\{b_n\}$ be summable sequences. Define a new sequence by $c_n = a_n + b_n$. Prove that the sequence $\{c_n\}$ is summable and that

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

(See exercise 1.5.2.)

14. Assume that the sequence $\{a_n\}$ is summable and that the sequence $\{b_n\}$ is not summable. Prove that the sequence given by $c_n = a_n + b_n$ is not summable. (See exercise 1.5.3.)

15. Evaluate the following sums:
    a.) $\sum_{n=1}^{\infty} \frac{1}{3^n}$
    b.) $\sum_{n=3}^{\infty} \frac{1}{3^n}$
    c.) $\sum_{n=1}^{\infty} \frac{1}{3^{n+2}}$
    d.) $\sum_{n=1}^{\infty} \frac{2}{3^n}$
    e.) $\sum_{n=1}^{\infty} \frac{9}{10^n}$
    f.) $\sum_{n=1}^{\infty} \left(\frac{1}{3^n} + \frac{3}{5^n} \right)$

16. A rubber ball bounces 1/3 of the height from which it falls. If it is dropped from 10 feet and allowed to continue bouncing, how far does it travel?

2.2   Series with Nonnegative Terms

In this section we will discuss tests for convergence of series with nonnegative terms. Similar results are true for sequences with nonpositive terms, but we won't dwell on that here. In the next section we will study series which may have some negative and some positive terms.

The key remark to begin this section is that if $\{a_n\}$ is a sequence of nonnegative terms, then the sequence of partial sums of $\{a_n\}$ is increasing. Applying the Property of Completeness, we get the following.

Proposition 2.2.1

If $\{a_n\}$ is a sequence of nonnegative terms then $\sum a_n$ converges if and only if the sequence of partial sums is bounded.

Proof: If the terms of a series are nonnegative then the sequence of partial sums is increasing. Thus, the Property of Completeness says that the sequence of partial sums converges if it is bounded. On the other hand, if the sequence of partial sums is not bounded, then surely it diverges to infinity.

$\square$

Proposition 2.2.2 (Comparison Test)

Let $\{a_n\}$ and $\{b_n\}$ be sequences of terms with $0 \le a_n \le b_n$. If $\sum b_n$ converges then $\sum a_n$ converges.

(Equivalently, if $\sum a_n$ diverges then $\sum b_n$ diverges.)

Proof: Let $\{s_n\}$ denote the sequence of partial sums for $\{a_n\}$ and let $\{t_n\}$ denote the sequence of partial sums for $\{b_n\}$, i.e.

$$s_n = a_1 + a_2 + ... + a_n$$

and

$$t_n = b_1 + b_2 + ... + b_n.$$

Then notice that since $0 \leq a_n \leq b_n$, we can conclude that $s_n \leq t_n$. (A formal proof of this would use induction.) Now, assume that $\sum b_n$ converges. Then, by proposition 2.2.1, we know that the sequence of partial sums $\{t_n\}$ is bounded. But since $s_n \leq t_n$ for all n, we see that the sequence $\{s_n\}$ is bounded as well. Again applying proposition 2.2.1, we see that $\sum a_n$ converges. $\qquad\square$

Remark:

Altering a finite number of terms of a series does not affect whether or not the series converges, thus the above tests for convergence or divergence are valid as long as the inequalities hold *eventually*. Thus, to apply the above theorem, it is enough to check that there is an $N \in \mathbb{N}$ so that the inequality $0 \leq a_n \leq b_n$ holds for all $n > N$.

Example 2.2.3

a.) Since we know that $\sum \frac{1}{n}$ diverges, we can conclude that if we have a sequence of terms $\{a_n\}$ with $\frac{1}{n} \leq a_n$ then $\sum a_n$ diverges too. For example, if $a_n = \frac{1}{\sqrt{n}}$, or if $a_n = \frac{n+1}{n^2+1}$.

b.) Similarly, since we know that $\sum \frac{1}{n^2}$ converges, we can conclude that if we have a sequence of terms $\{a_n\}$ with $a_n \leq \frac{1}{n^2}$ then $\sum a_n$ converges too. For example, if $a_n = \frac{1}{n^2+1}$ or $a_n = \frac{1}{n^3}$.

As in the example, the above test shows easily that the series $\sum \frac{1}{n^2+1}$ converges by comparison to $\sum \frac{1}{n^2}$. On the other hand, it can be pretty tricky to try to use the comparison test to determine the convergence or divergence of the series $\sum \frac{1}{n^2-1}$. The following refinement of the comparison test makes this much easier.

Proposition 2.2.4 (Limit–Comparison Test)

Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative terms.

a.) If $\sum b_n$ converges and the sequence $\{\frac{a_n}{b_n}\}$ has a finite limit, then $\sum a_n$ also converges.

b.) If $\sum b_n$ diverges and the sequence $\{\frac{a_n}{b_n}\}$ either has a nonzero limit or diverges to infinity, then $\sum a_n$ also diverges.

c.) If the sequence $\{\frac{a_n}{b_n}\}$ has a nonzero finite limit then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Proof:

a.) Assume $\lim \frac{a_n}{b_n} = L$. Then we can choose an $N \in \mathbb{N}$ so that $\frac{a_n}{b_n} < L + 1$, for all $n > N$. That is, $a_n < b_n(L + 1)$ for $n > N$. Now, since we

assume that $\sum b_n$ converges, it follows that $\sum b_n(L+1)$ also converges (see exercise 2.1.12). Thus the comparison test applies to prove that $\sum a_n$ is also convergent.

b.) If the sequence $\{\frac{a_n}{b_n}\}$ has a nonzero limit or diverges to infinity, then the sequence $\{\frac{b_n}{a_n}\}$ has a finite limit. Applying part a.) we see that if $\sum a_n$ converges then so must $\sum b_n$. Hence, if we know that $\sum b_n$ is divergent, we conclude that $\sum a_n$ could not be convergent, i.e., it diverges.

c.) If the sequence $\{\frac{a_n}{b_n}\}$ has a nonzero finite limit, then so does the sequence $\{\frac{b_n}{a_n}\}$. Applying part a.) we see that if either $\sum a_n$ or $\sum b_n$ converges, then so must the other. Applying part b.), we see that if either diverges, then so does the other.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Example 2.2.5

It is now easy to check that $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ is convergent by letting $a_n = \frac{1}{n^2-1}$ and $b_n = \frac{1}{n^2}$ and noting that $\lim a_n/b_n = 1$.

For the final convergence test of this section, we need to recall the concept of *improper integrals* from Calculus. Let f be a continuous, nonnegative function defined on the nonnegative real numbers, i.e. $f : \mathbb{R}^+ \to \mathbb{R}^+$. Recall that we say that the *improper integral* $\int_1^{\infty} f(x)\,dx$ *converges* if $\lim_{b\to\infty} \int_1^b f(x)\,dx$ converges. Recall also that we say that the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *decreasing* if for each $x, y \in \mathbb{R}^+$ with $x \leq y$ we have $f(x) \geq f(y)$.

Proposition 2.2.6 (Integral Comparison Test)

Let f be a nonnegative, decreasing function defined on the set of positive real numbers. Let $a_n = f(n)$. Then the series $\sum a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Proof: Define the sequence $b_n = \int_n^{n+1} f(x)\,dx$. Notice that $\int_1^{n+1} f(x)\,dx = \sum_{j=1}^n b_j$. Since the function f is nonnegative, it follows that the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent if and only if the series $\sum_1^{\infty} b_j$ is convergent. Now notice that since f is decreasing, we have that for each $j \in \mathbb{N}$,

$$ f(j+1) \leq \int_j^{j+1} f(x)\,dx \leq f(j), $$

i.e., $a_{j+1} \le b_j \le a_j$. From the first of these inequalities we see that if $\sum b_j$ converges then so must $\sum a_j$ and from the second of these inqualities we see that if $\sum b_j$ diverges, then so must $\sum a_j$.

$\square$

Example 2.2.7

Using the fact that $\int_1^\infty x^{-r}\, dx$ converges if and only if $r > 1$, we conclude that $\sum n^{-r}$ converges if and only if $r > 1$. In particular, $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^5}$, and $\sum \frac{1}{n^{3/2}}$ all converge, while $\sum \frac{1}{n}$, and $\sum \frac{1}{n^{1/3}}$ diverge.

Example 2.2.8

Since $\int_2^b \frac{1}{x \ln x}\, dx = \ln(\ln x)|_2^b = \ln(\ln b) - \ln(\ln 2)$. We can conclude that $\int_2^\infty \frac{1}{x \ln x}\, dx$ diverges, and thus so does $\sum \frac{1}{n \ln n}$.

Notice that we cannot get this divergence by comparing to the series whose terms are $\frac{1}{n}$ since $\lim \frac{1}{n \ln n} / \frac{1}{n} = \lim \frac{1}{\ln n} = 0$ and so the limit – comparison test does not apply.

Example 2.2.9 (Euler's Constant)

Returning to the divergence of $\sum \dfrac{1}{n}$, the integral test tells us that as n goes to infinity, the partial sum $\displaystyle\sum_{k=1}^n \dfrac{1}{k}$ diverges to infinity at roughly the same rate as $\int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1)$. Thus, it makes sense to study the limit of the difference $\displaystyle\sum_{k=1}^n \dfrac{1}{k} - \ln(n+1)$, which we will denote γ_n. Notice then that

$$\gamma_n = \sum_{k=1}^n \frac{1}{k} - \ln(n+1)$$

$$= \sum_{k=1}^n \int_k^{k+1} \left(\frac{1}{k} - \frac{1}{x} \right) dx$$

$$= \sum_{k=1}^n \int_k^{k+1} \frac{x-k}{xk}\, dx.$$

From this it is clear that the sequence $\{\gamma_n\}$ is strictly increasing. Furthermore, when $k \le x \le k+1$ we have that

$$\frac{x-k}{xk} \le \frac{1}{xk} \le \frac{1}{(x-1)^2}$$

where the last inequality comes from the facts that $0 \leq (x-1) \leq k$ and $0 \leq (x-1) \leq x$ and so $(x-1)^2 \leq xk$. Thus we see that the sequence $\{\gamma_n\}$ is bounded above since

$$\gamma_n = 1 - \ln(2) + \sum_{k=2}^{n} \int_{k}^{k+1} \frac{x-k}{xk}\, dx$$

$$\leq 1 - \ln(2) + \sum_{k=2}^{n} \int_{k}^{k+1} \frac{1}{(x-1)^2}\, dx$$

$$\leq 1 + \int_{2}^{\infty} \frac{dx}{(x-1)^2}$$

$$= 2.$$

Since this sequence is increasing and bounded above, it must have a limit. The limit $\gamma = \lim \gamma_n$ is called *Euler's constant*. Approximately $\gamma \approx 0.57721566$; it is not known whether or not γ is rational.

EXERCISES 2.2

1. Determine whether the series converges or diverges. Give reasons.

    a.) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ b.) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

    c.) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ d.) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$

    e.) $\sum_{n=1}^{\infty} \left(\frac{1}{n^2+1}\right)^n$ f.) $\sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^n$

    g.) $\sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^{-n}$ h.) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

    i.) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2+1}$ j.) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/2}$

    k.) $\sum_{n=1}^{\infty} n 2^{-n}$ l.) $\sum_{n=1}^{\infty} (n+1)^{-1/5}$

2. Let $\sum a_n$ be a series of nonnegative terms. Show that if $\sum a_n$ converges then $\sum a_n^2$ converges also.

3. Give an example of a divergent series whose sequence of partial sums is bounded.

4. For which real numbers r does the series $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^r}$ converge?

5. For $r \in \mathbb{R}$ define $\gamma_n(r)$ by

$$\gamma_n(r) = \sum_{k=1}^{n} \frac{1}{k^r} - \int_{1}^{n+1} \frac{1}{x^r}\,dx.$$

For which values of r does $\lim_{n\to\infty} \gamma_n(r)$ exist?

6. a.) Show that any series of the form $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$, with $d_n \in \{0,1,2,3,4,5,6,7,8,9\}$, is convergent. (Hint: $\frac{d_n}{10^n} \le \frac{9}{10^n}$.)

b.) Given $x \in [0,1)$ define d_n recursively as follows: First, notice that $0 \le 10x < 10$ so we can choose an integer $d_1 \in \{0,1,2,3,4,5,6,7,8,9\}$ with

$$d_1 \le 10x < d_1 + 1.$$

Notice that this is equivalent to

$$0 \le 10(x - \frac{d_1}{10}) < 1.$$

Now, as in the first step, since $10(x - \frac{d_1}{10})$ is in the interval $[0,1)$, we can choose $d_2 \in \{0,1,2,3,4,5,6,7,8,9\}$ so that

$$d_2 \le 10(10(x - \frac{d_1}{10})) < d_2 + 1.$$

This can be rewritten as

$$0 \le 100(x - \frac{d_1}{10} - \frac{d_2}{100}) < 1.$$

Continuing this process, show by induction that we can find $d_n \in \{0,1,2,3,4,5,6,7,8,9\}$ so that

$$0 \le 10^n(x - \frac{d_1}{10} - \frac{d_2}{100} - \cdots - \frac{d_n}{10^n}) < 1.$$

c.) With the d_n's defined as in part b.) show that the series $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ converges to x.

Remark: This exercise shows that all real numbers have a decimal expansion.

7. Show that

$$\lim_{n \to \infty} \sum_{k=n}^{2n} \frac{1}{k} = \ln 2.$$

(Hint: look at $\gamma_{2n} - \gamma_{n-1}$.)

2.3   Series with Terms of Both Signs

Of course if all of the terms of a series $\sum a_n$ are nonpositive we can apply the results from the last section to the series $\sum -a_n$ and then multiply by a negative sign. Also, if all of the terms are eventually nonnegative (or eventually nonpositive) then convergence is determined by the convergence of the series $\sum |a_n|$ since convergence of a series does not change if we alter a finite number of the terms. In this chapter we will be concerned mostly with series that have infinitely many positive terms and infinitely many negative terms.

An important test for convergence of such a series is the following

Proposition 2.3.1

If the series $\sum |a_n|$ converges, then so does the series $\sum a_n$.

Proof: We will use the Cauchy criterion to prove convergence of the partial sums. Let

$$s_n = \sum_{j=1}^{n} a_j, \quad \text{and} \quad t_n = \sum_{j=1}^{n} |a_j|.$$

Since $\sum |a_n|$ converges we know that if we are given an $\epsilon > 0$ we can find an $N \in \mathbb{N}$ so that $|t_n - t_m| < \epsilon$ whenever $n, m > N$. But notice that if $n > m$ then

$$
\begin{aligned}
|s_n - s_m| &= |a_{m+1} + a_{m+2} + ... + a_n| \\
&\leq |a_{m+1}| + |a_{m+2}| + ... + |a_n| \\
&= t_n - t_m \\
&= |t_n - t_m|
\end{aligned}
$$

by the triangle inequality. A similar remark is true if $m > n$, thus if $n, m > N$ we have

$$|s_n - s_m| \le |t_n - t_m| < \epsilon$$

proving that the sequence of partial sums $\{s_n\}$ is a Cauchy sequence. Therefore $\sum a_n$ converges.

$\square$

Example 2.3.2
 Since we know that the series $\sum \frac{1}{n^2}$ converges, it follows that the series $\sum \frac{(-1)^n}{n^2}$ converges also.

Definition 2.3.3
 If the series $\sum |a_n|$ converges then the series $\sum a_n$ is said to be *absolutely convergent*.

Of course, Proposition 2.3.1 says that absolute convergence implies convergence. The converse of this statement is false, as we see in the following example.

Example 2.3.4
 The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent but not absolutely convergent.

We have already shown that this series is not absolutely convergent since we know that $\sum \frac{1}{n}$ diverges to infinity. The convergence follows from a more general statement:

Proposition 2.3.5 (Alternating Series Test)
 If the sequence $\{b_n\}$ is decreasing and limits to 0, then the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

Proof: Let $s_n = \sum_{j=1}^{n} (-1)^j b_j$ be the nth partial sum of the series. Notice that the subsequence given by $c_n = s_{2n}$ is decreasing since

$$\begin{aligned} c_n - c_{n-1} &= s_{2n} - s_{2n-2} \\ &= (-1)^{2n} b_{2n} + (-1)^{2n-1} b_{2n-1} \\ &= b_{2n} - b_{2n-1} \\ &\le 0. \end{aligned}$$

Furthermore, since $b_{2k} - b_{2k+1} \ge 0$ for all k, it can be shown by induction that $c_n \ge -b_1 + b_{2n}$ and hence the sequence $\{c_n\}$ is decreasing and bounded below

by $-b_1$. Thus the subsequence $\{s_{2n}\}$ converges to some limit which we call L. Now considering the subsequence given by the odd terms $\{s_{2n+1}\}$ we see that $s_{2n+1} = s_{2n} - b_{2n+1}$ and so we have $\lim s_{2n+1} = \lim s_{2n} - \lim b_{2n+1} = L$ as well. From this one can prove that $\lim s_n = L$ too (see exercise 2.3.1).

$\square$

Definition 2.3.6

A series which converges but does not converge absolutely is called *conditionally convergent*.

Remark: A convergent series is either absolutely convergent or conditionally convergent, but it can not be both. When determining if a given series $\sum a_n$ is absolutely convergent, conditionally convergent, or divergent, it is usually best to consider first the series of absolute values, $\sum |a_n|$, whose terms are nonnegative. If this series converges then the original series is absolutely convergent and you are done. On the other hand, if the series $\sum |a_n|$ diverges, then you must still check to see if the original series converges or diverges. In this case, if it converges then the convergence is conditional. Notice that we have many good tests for the convergence of $\sum |a_n|$, but only a few tests for the convergence of $\sum a_n$ when $\sum |a_n|$ diverges.

Example 2.3.7

a.) Let's determine whether the series $\sum \dfrac{(-1)^n(3n+2)}{n^3+1}$ converges absolutely, converges conditionally, or diverges. As suggested in the above remark we first check for absolute convergence. Since the absolute values of the terms, $\dfrac{3n+2}{n^3+1}$, limit to zero at about the rate of $\dfrac{1}{n^2}$ we are led to try the limit – comparison test. Since $\lim \dfrac{\left(\frac{3n+2}{n^3+1}\right)}{\left(\frac{1}{n^2}\right)} = 3$, we conclude that the series $\sum \dfrac{3n+2}{n^3+1}$ converges. Thus the series $\sum \dfrac{(-1)^n(3n+2)}{n^3+1}$ is absolutely convergent.

b.) Now let's determine whether the series $\sum \dfrac{(-1)^n(2n+1)}{n^2+1}$ converges absolutely, converges conditionally, or diverges. As above, we first check for absolute convergence. Since the absolute values of the terms, $\dfrac{2n+1}{n^2+1}$, limit to zero at about the rate of $\dfrac{1}{n}$ we are again led to try the limit – comparison test.

Since $\lim \dfrac{\left(\frac{2n+1}{n^2+1}\right)}{\left(\frac{1}{n}\right)} = 2$, we conclude that the series $\sum \dfrac{2n+1}{n^2+1}$ diverges, and so the series $\sum \dfrac{(-1)^n(2n+1)}{n^2+1}$ is not absolutely convergent. On the other hand, we have already noticed that $\lim \dfrac{2n+1}{n^2+1} = 0$ so we need only check that this sequence is decreasing to use Proposition 2.3.5 to see that the series $\sum \dfrac{(-1)^n(2n+1)}{n^2+1}$ converges conditionally. However, with $a_n = \dfrac{2n+1}{n^2+1}$ we see that, for $n \in \mathbb{N}$

$$a_n - a_{n+1} = \frac{2n+1}{n^2+1} - \frac{2(n+1)+1}{(n+1)^2+1}$$

$$= \frac{(2n+1)(n^2+2n+2) - (2n+3)(n^2+1)}{(2n+1)((n+1)^2+1)}$$

$$= \frac{(2n^3+5n^2+6n+2) - (2n^3+3n^2+2n+3)}{(2n+1)((n+1)^2+1)}$$

$$= \frac{2n^2+4n-1}{(2n+1)((n+1)^2+1)}$$

$$> 0.$$

Of course, if a sequence has only finitely many negative terms, then it is convergent if and only if it is absolutely convergent. Thus, a conditionally convergent sequence must have infinitely many negative terms. Similarly, it must also have infinitely many positive terms. That is, *a conditionally convergent sequence must have infinitely many positive terms and infinitely many negative terms.* Suppose now that $\sum a_n$ is conditionally convergent. We can define two new series with nonnegative terms $\{b_n\}$ and $\{c_n\}$ given by

$$b_n = \max\{a_n, 0\} = \frac{1}{2}(|a_n| + a_n)$$

$$c_n = -\min\{a_n, 0\} = \frac{1}{2}(|a_n| - a_n).$$

That is, $b_n = a_n$ when $a_n \geq 0$, and is zero otherwise, while $c_n = -a_n$ when $a_n \leq 0$ and is zero otherwise. Notice that $b_n - c_n = a_n$ and $b_n + c_n = |a_n|$.
Remark: The sequences $\{b_n\}$ and $\{c_n\}$ are usually **not** subsequences of $\{a_n\}$.

Proposition 2.3.8

Assume that the series $\sum a_n$ is conditionally convergent and define the sequences $\{b_n\}$ and $\{c_n\}$ as above. Then both $\sum b_n$ and $\sum c_n$ diverge to infinity.

Proof: To the contrary, suppose that either $\sum b_n$ or $\sum c_n$ converges. Let's look at the case where we suppose that $\sum b_n$ converges. Then since $\sum a_n$ converges, it follows that $\sum (b_n - a_n)$ also converges. But $b_n - a_n = c_n$, so we can conclude that $\sum c_n$ converges. Similarly if we begin with the assumption that $\sum c_n$ converges, we can prove that $\sum b_n$ converges too. That is, if either of $\sum b_n$ or $\sum c_n$ converges, then the other one converges too. However, if both $\sum b_n$ and $\sum c_n$ converge, then $\sum (b_n + c_n)$ converges. But $b_n + c_n = |a_n|$, so the above implies that $\sum a_n$ is absolutely convergent.

$\square$

We will study an amazing property of conditionally convergent sequences at the end of this section but before that, we present two important tests for absolute convergence.

Proposition 2.3.9 (Ratio Test)
Suppose that $\lim \frac{|a_{n+1}|}{|a_n|} = L$
a.) If $L < 1$, then the series $\sum a_n$ converges absolutely.
b.) If $L > 1$, then the series $\sum a_n$ diverges.
(Furthermore, if $\lim \frac{|a_{n+1}|}{|a_n|} = \infty$ then the series $\sum a_n$ diverges.)

The proof of this proposition is outlined in exercise 2.3.4. Similar techniques allow us to prove the *Root Test*, (see exercise 2.3.5).

Proposition 2.3.10 (Root Test)
Suppose that $\lim (|a_n|)^{\frac{1}{n}} = L$.
a.) If $L < 1$, then the series $\sum a_n$ converges absolutely.
b.) If $L > 1$, then the series $\sum a_n$ diverges.
(Furthermore, if $\lim (|a_n|)^{\frac{1}{n}} = \infty$ then $\sum a_n$ diverges.)

Remark: Notice that the above tests give *no information* if the limits in the hypotheses don't exist or are equal to 1.

Example 2.3.11

a.) Applying the ratio test to the series $\sum \frac{n^2}{2^n}$ we see that

$$
\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} &= \frac{(n+1)^2/2^{n+1}}{n^2/2^n} \\
&= \frac{2^n(n+1)^2}{2^{n+1}n^2} \\
&= \frac{(n+1)^2}{2n^2}
\end{aligned}
$$

And so $\lim \frac{|a_{n+1}|}{|a_n|} = 1/2$. Since this is less than 1, we conclude that the series converges.

b.) Applying the ratio test to the series $\sum \frac{n^2 a^n}{3^n}$ we see that

$$
\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} &= \frac{|(n+1)^2 a^{n+1}/3^{n+1}|}{|n^2 a^n/3^n|} \\
&= \frac{|a|(n+1)^2}{3n^2}.
\end{aligned}
$$

And so $\lim \frac{|a_{n+1}|}{|a_n|} = |a|/3$. Thus we see that this series converges if $-3 < a < 3$, and it diverges if $|a| > 3$. the ratio test tells us nothing about the cases that $a = 3$ or $a = -3$, but we can easily see that the series $\sum n^2$ and $\sum (-1)^n n^2$ are divergent since the terms do not limit to zero.

c.) Applying the root test to the series $\sum \frac{1}{n^n}$ we see that $|a_n|^{1/n} = (\frac{1}{n^n})^{1/n} = \frac{1}{n}$. Since $\lim \frac{1}{n} = 0$, we conclude that the series converges.

Rearrangements

We know that the series $\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n}$ is conditionally convergent to some real number α although we have no idea what that number is.[1]   So we have

$$
\alpha = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots
$$

Now consider the "rearrangement" of this series given by

$$
1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots
$$

---

[1]You can show in exercise 12 that $\alpha = \ln 2$.

where we interweave two of the negative terms between each pair of positive terms. By regrouping this sum slightly, we see that this gives

$$
\begin{aligned}
(1 - \frac{1}{2}) &- \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \ldots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \ldots \\
&= \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots)
\end{aligned}
$$

i.e. half of the original sum. Although this may seem surprising, we have an even more surprising result.

Proposition 2.3.12

If the series $\sum a_n$ is conditionally convergent, then, given any $\alpha \in \mathbb{R}$, there is some rearrangement $\{b_n\}$ of the terms $\{a_n\}$ so that

$$
\sum_{n=1}^{\infty} b_n = \alpha.
$$

Proof: Assume $\sum a_n$ is conditionally convergent and let $\alpha \in \mathbb{R}$ be given. Let $\{p_n\}$ denote the subsequence given by the positive terms of $\{a_n\}$ and let $\{q_n\}$ denote the subsequence given by the negative terms of $\{a_n\}$. By proposition 2.3.8 we know that both $\sum p_n$ and $\sum -q_n$ diverge to infinity. Thus we can find $N_1 \in \mathbb{N}$ so that

$$
p_1 + p_2 + \ldots + p_{N_1 - 1} \leq \alpha < p_1 + p_2 + \ldots + p_{N_1}.
$$

Let

$$
b_1 = p_1, \ b_2 = p_2, \ \ldots, \ b_{N_1} = p_{N_1}.
$$

Also let $S_1 = p_1 + p_2 + \ldots + p_{N_1}$ and notice that

$$
S_1 - \alpha < p_{N_1}.
$$

Now the q_j's are negative and sum to $-\infty$ so we can find $M_1 \in \mathbb{N}$ so that

$$
S_1 + q_1 + q_2 + \ldots + q_{M_1} < \alpha \leq S_1 + q_1 + q_2 + \ldots + q_{M_1 - 1}.
$$

Let

$$
b_{N_1 + 1} = q_1, \ b_{N_1 + 2} = q_2, \ \ldots, \ b_{N_1 + M_1} = q_{M_1},
$$

and let $T_1 = S_1 + q_1 + q_2 + ... + q_{M_1}$. We have

$$\alpha - T_1 < -q_{M_1}.$$

We can continue this process forever, defining the sequence $S_1, T_1, S_2, T_2, ...$ of partial sums of a rearrangement of $\{a_n\}$ with $|S_n - \alpha| < p_{N_n}$ and $|T_n - \alpha| < -q_{M_n}$ where $\{M_n\}$ and $\{N_n\}$ are sequences of natural numbers limiting to infinity. Since the original series $\sum a_n$ converges, we know that $\lim p_n = \lim q_n = 0$, so we can conclude that these partial sums limit to α. $\square$

On the other hand, if $\sum a_n$ converges absolutely, rearrangements won't effect the sum:

Proposition 2.3.13
If the series $\sum a_n$ is absolutely convergent, then for any rearrangement $\{b_n\}$ of the terms $\{a_n\}$, we have $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

Proof: Assume that $\sum |a_n|$ converges and let $\{b_n\}$ be a rearrangement of $\{a_n\}$. For simplicity let A denote the sum $A = \sum_{n=1}^{\infty} a_n$ and let $\tilde{A}$ denote the sum $\tilde{A} = \sum_{n=1}^{\infty} |a_n|$. Then, given $\epsilon > 0$, there is a natural number N so that

$$\tilde{A} - (|a_1| + |a_2| + \cdots + |a_N|) < \epsilon.$$

Another way to write this is

$$|a_{N+1}| + |a_{N+2}| + \cdots < \epsilon.$$

Now pick M so large that all of the terms $a_1, a_2, \ldots, a_N$ appear among the terms $b_1, b_2, \ldots, b_M$ of the rearranged series. For $m > M$, let s_m denote the partial sum $s_m = b_1 = b_2 + \cdots + b_m$ and notice that $A - s_m$ is a sum of the terms a_n with $n > N$ and with a finite number of other terms missing. Thus, using exercise 9, we get

$$|A - s_m| \le |a_{N+1}| + |a_{N+2}| + \cdots \le \epsilon.$$

Finally, to prove that the series $\sum b_n$ converges absolutely, just note that the series $\{|b_n|\}$ is a rearrangement of the series $\{|a_n|\}$ so the above result shows that $\sum |b_n| = \sum |a_n|$.

$\square$

EXERCISES 2.3

1. Let $\{s_n\}$ be a sequence and let $\{c_n\}$ and $\{d_n\}$ be the subsequences given by $c_n = s_{2n}$ and $d_n = s_{2n+1}$. Assume that $\lim c_n = L$ and that $\lim d_n = L$. Prove that $\lim s_n = L$.

2. What does the ratio test tell you about the following series?

    a.) $\sum \frac{2^n}{n!}$

    b.) $\sum \frac{n!}{2^n}$

    c.) $\sum \frac{n}{3^n}$

    d.) $\sum \frac{1}{n^2}$

3. Which of the following series are absolutely convergent, conditionally convergent, or divergent?

    a.) $\sum \frac{(-1)^n}{n^2}$

    b.) $\sum \frac{(-1)^n}{\sqrt{n}}$

    c.) $\sum \frac{(-1)^n n^2}{n!}$

    d.) $\sum \frac{(-1)^n (n^2+3)}{3n^2+1}$

    e.) $\sum \frac{(-1)^n}{n \ln n}$

    f.) $\sum \frac{(-1)^n n}{\sqrt{2^n+1}}$

    g.) $\sum n(-\frac{2}{5})^n$

4. We prove the ratio test in this exercise.

    a.) Assume $L < 1$ and fix r with $L < r < 1$. Show that for sufficiently large n, $|a_{n+1}| \leq r|a_n|$.

    b.) Explain why we can assume $|a_{n+1}| \leq r|a_n|$ for all $n \in \mathbb{N}$.

    c.) Using part b.), show by induction that $|a_n| \leq r^{n-1}|a_1|$ for all n.

    d.) Apply the comparison test to show that $\sum |a_n|$ converges.

    e.) Now assume that $L > 1$ and show that $|a_{n+1}| > |a_n|$ for sufficiently large n. Show that this implies that $\lim a_n$ cannot be 0.

5. Prove the root test by an argument similar to that outlined in problem 4.

6. For which values of $a \in \mathbb{R}$ do the following series converge?

   a.) $\sum \frac{a^n}{2^n}$

   b.) $\sum \frac{na^n}{2^n}$

   c.) $\sum (\frac{4}{a^2})^n$

   d.) $\sum \frac{(-a)^n}{n!}$

   e.) $\sum \frac{(-a)^n}{n^2}$

7. Prove that if $\sum a_n$ diverges then $\sum |a_n|$ diverges.

8. Here is an alternate proof of Proposition 2.3.1. Assume $\sum |a_n|$ converges and let $b_n = a_n + |a_n|$. Notice that $0 \le b_n \le 2|a_n|$ so, by the comparison test, $\sum b_n$ converges. Use this and the fact that $\sum |a_n|$ converges to prove that $\sum a_n$ converges.

9. Assume that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that

$$|\sum_{n=1}^{\infty} a_n| \le \sum_{n=1}^{\infty} |a_n|.$$

10. Write down the first 12 terms of a rearrangement of the alternating harmonic series which converges to 1.

11. Find (or at least describe a method of finding) a rearrangement of the alternating harmonic series that diverges to $+\infty$.

12. One can use the convergence of the sequence $\{\gamma_n\}$ given in Example 2.2.9 to evaluate the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ as follows:

    a.) Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$ denote the nth partial sum of the harmonic series and show that
    $$1 - \frac{1}{2} + \frac{1}{3} - ... + \frac{1}{2n-1} - \frac{1}{2n} = s_{2n} - s_n.$$

b.) Use the fact that $s_n = \gamma_n + \ln(n+1)$ to write

$$1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{2n-1} - \frac{1}{2n} = (\gamma_{2n} - \gamma_n) + \ln\left(\frac{2n+1}{n+1}\right).$$

c.) Taking the limit as n goes to infinity, conclude that $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2$.

13. Let $\{a_n\}$ and $\{b_n\}$ be sequences and define $c_n = \sum_{k=1}^{n} a_k b_{n-k+1}$. The series $\sum_{n=1}^{\infty} c_n$ is called the *Cauchy Product* of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. Let $a_n = b_n = (-1)^n / \sqrt{n}$ and show that $|c_n| \geq 1$ so that although $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, we have $\sum_{n=1}^{\infty} c_n$ diverges.

   Note: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely, it can be proven that $\sum_{n=1}^{\infty} c_n$ converges absolutely and that $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$.

2.4 Power Series

A *power series* is an expression of the form

$$\sum a_n x^n$$

where x is a real variable. We can think of this as defining a function whose natural domain of definition is the set of real numbers, x, such that the sum converges. This set is called the *domain of convergence.*

Definition 2.4.1

The *domain of convergence* of the power series $\sum a_n x^n$ is given by $\{x \in \mathbb{R} \mid \sum a_n x^n$ converges$\}$.

Example 2.4.2

a.) The geometric series $\sum x^n$ has domain of convergence given by the interval (-1,1). In fact, for $x \in (-1, 1)$ we know that $\sum_{n=1}^{\infty} x^n = x/(1 - x)$. The function $f(x) = x/(1-x)$ is called a *closed form* expression for $\sum_{n=1}^{\infty} x^n$.

b.) The series $\sum \frac{x^n}{n}$ has domain of convergence given by [-1,1). To see this, we first apply the ratio test to the terms $\frac{x^n}{n}$: $\lim |\frac{x^{n+1}}{n+1}|/|\frac{x^n}{n}| = \lim \frac{|nx|}{n+1} = |x|$. So the series converges if $|x| < 1$ and diverges if $|x| > 1$. Of course the test tells us nothing if $|x| = 1$ so we check those two cases separately. If $x = 1$, the series becomes the harmonic series which diverges, and if $x = -1$ the series becomes the alternating harmonic series, which converges. Thus the domain of convergence is [-1,1).

It is no coincidence that the domains of convergence for these examples are both intervals.

Proposition 2.4.3

Suppose that the power series $\sum a_n x^n$ converges for $x = c$. Then it converges absolutely for all x such that $|x| < |c|$.

Proof: Since the sequence of terms $\{a_n c^n\}$ of the convergent series $\sum a_n c^n$, converges to zero, we know that this sequence must be bounded. I.e. there is some number $K > 0$ so that $|a_n c^n| < K$, for all n. Now for $|x| < |c|$ set $d = |x|/|c|$. Then for each n, we have $|a_n x^n| = |a_n x^n c^n / c^n| = |a_n c^n d^n| = |a_n c^n| d^n < K d^n$. But $0 < d < 1$ so $\sum K d^n$ is a convergent geometric series, thus, by comparison we conclude that $\sum |a_n x^n|$ converges.

$\square$

Corollary 2.4.4

The domain of convergence for a power series $\sum a_n x^n$ is an interval centered at zero. Thus the domain of convergence has one of the following forms: $[-R, R]$, $(-R, R]$, $[-R, R)$, $(-R, R)$ with $R \geq 0$, or all of $\mathbb{R}$.

Another way of stating this result is that if the power series $\sum a_n x^n$ does not converge for all real numbers, then there is some nonnegative real number R so that the power series converges for all real x with $|x| < R$ and diverges for all real x with $|x| > R$.

Definition 2.4.5

If R is a nonnegative real number such that the power series $\sum a_n x^n$ converges for all real x with $|x| < R$ and diverges for all real x with $|x| > R$ then R is called the *radius of convergence* for the power series. If the power series converges for all real numbers, the radius of convergence is said to be *infinite*.

Remark: A power series *centered at a* is a series of the form

$$\sum a_n (x - a)^n$$

where a represents some fixed real number. Of course this can be viewed as a horizontal shift of the power series $\sum a_n x^n$, so if the latter series has radius of convergence given by R then so does the former series. On the other hand, if the domain of convergence of the latter series is, say, $(-R, R)$ then the domain of convergence of the former series is $(-R + a, R + a)$.

Example 2.4.6

Since we know that the power series $\sum x^n/n$ has domain of convergence $[-1, 1)$ we can conclude that the series $\sum (x + 2)^n/n$ has domain of convergence given by $[-3, -1)$.

The above shifting technique can be thought of as the *composition* of the function defined by the power series $f(x) = \sum a_n x^n$ with the function $g(x) = x - a$. Composition with other simple functions can lead to similar generalizations.

Example 2.4.7

The power series $h(x) = \sum x^{2n}/2^n$ can be thought of as the composition of the power series $f(x) = \sum x^n$ with the function $g(x) = x^2/2$. Since we know that the domain of convergence for $f(x)$ is $(-1, 1)$ we can conclude

that $h(x)$ converges exactly when $-1 < x^2/2 < 1$. Thus, the domain of convergence for $h(x)$ must be $(-\sqrt{2}, \sqrt{2})$.

Finally we remark that although power series define functions on their domains of convergence, we generally cannot find closed form expressions for these functions. In the next chapter we will extend the list of power series for which we can determine closed form expressions, but for now we can do so only for the geometric series and series that are related to the geometric series through some manipulation.

Example 2.4.8
a.) Since the power series $h(x) = \sum_{n=1}^{\infty} (x-2)^n/2^n$ can be thought of as the composition of $f(x) = \sum_{n=1}^{\infty} x^n$ with $g(x) = (x-2)/2$ we know that

$$h(x) = \frac{(x-2)/2}{1 - (x-2)/2} = \frac{x-2}{4-x}$$

when $-1 < (x-2)/2 < 1$.

b.) Since $\sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$ we have $\sum_{n=0}^{\infty} x^n = 1 + \dfrac{x}{1-x} = \dfrac{1}{1-x}$.

EXERCISES 2.4

1. Compute the domains of convergence for the following power series.

a.) $\displaystyle\sum nx^n$

b.) $\displaystyle\sum \frac{x^n}{3^n}$

c.) $\displaystyle\sum \frac{x^n}{n!}$

d.) $\displaystyle\sum 3^n(x-1)^n$

e.) $\displaystyle\sum \frac{n!x^n}{2^n}$

f.) $\displaystyle\sum \frac{3x^n}{n^2}$

g.) $\displaystyle\sum \frac{(3x)^n}{n^2}$

h.) $\displaystyle\sum \frac{(x-5)^n}{n^4}$

i.) $\displaystyle\sum \frac{x^{3n}}{2^n}$

j.) $\displaystyle\sum \frac{(x-2)^{2n}}{3^n}$

k.) $\displaystyle\sum \frac{(-3)^n x^n}{\sqrt{n+1}}$

l.) $\displaystyle\sum \frac{n(x+2)^n}{3^{n+1}}$

2.   Find closed forms for the following power series.

a.) $\displaystyle\sum_{n=3}^{\infty} x^n$

b.) $\displaystyle\sum_{n=0}^{\infty} 2x^n$

c.) $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

d.) $\displaystyle\sum_{n=1}^{\infty} (2x-1)^n$

e.) $\displaystyle\sum_{n=1}^{\infty} x^{2n}$

3.   Find <u>a</u> *power series* representation for the following functions, be sure to indicate the *domain of convergence* and *radius of convergence* in each case:

$$(a) \ f(x) = \frac{1}{2+x^2} \qquad\qquad (b) \ g(x) = \frac{4}{x+4}$$

4.   a.) Find a power series which converges to $f(x) = \frac{1}{5-x}$ on the interval $(-5,5)$. (Hint: $5 - x = 5(1 - x/5)$.)

   b.) Find a power series which converges to $g(x) = \frac{1}{5+x}$ on the interval $(-5,5)$.

5.   Find a power series which converges to $f(x) = \frac{1}{5-x}$ on the interval $(3,5)$. (Hint: $5 - x = 1 - (x - 4)$.)

6.   Find a power series which converges to $f(x) = \frac{1}{(x-1)(x-2)}$ on the interval $(-1,1)$. (Hint: partial fractions.)

7.   Let $\{F_n\}$ be the Fibonacci sequence, so $F_0 = 1, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Let F(x) be the function given by

$$F(x) = \sum_{n=0}^{\infty} F_n x^n$$
$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots$$

$F(x)$ is called the *generating function* for $\{F_n\}$.

a.) By writing out $F(x) = 1 + x + \sum_{n=2}^{\infty}(F_{n-1} + F_{n-2})x^n$ show that $(1 - x - x^2)F(x) = 1$.

b.) Expand $\frac{1}{1-x-x^2}$ by partial fractions to write $F(x)$ in the form

$$F(x) = \frac{A}{\alpha - x} + \frac{B}{\beta - x}$$

for some numbers A and B, where $\alpha = (-1+\sqrt{5})/2$ and $\beta = (-1-\sqrt{5})/2$ are the roots of $1 - x - x^2 = 0$.

c.) By combining the power series for $\frac{A}{\alpha-x}$ and $\frac{B}{\beta-x}$, (and using that $1/\alpha = -\beta$) show that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

d.) Check that this formula really works for $n = 0, 1$, and 2.

8. (From the William Lowell Putnam Mathematical Competition, 1999)

   Consider the power series expantion

   $$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

   Prove that, for each integer $n \geq 0$, there is an integer m such that

   $$a_n^2 + a_{n+1}^2 = a_m.$$

Chapter 3

Sequences and Series of Functions

3.1 Uniform Convergence

In the last section we began discussing series whose terms were functions of a variable x. Now we discuss series and sequences of functions in more generality.

A *sequence of (real) functions* is a sequence $\{f_n\}$, where each term is a real valued function of a real variable. Of course many common functions do not have the whole real line as their domain, so each term f_n also has associated to it some domain $U_n \subset \mathbb{R}$. We are interested in studying the sequence on sets A on which all of the terms are defined, hence A must be contained in the intersection of the U_n's. So in general, when we discuss a sequence of functions on some set A, it is implied that all of the terms of the sequence are defined on A. Notice that for each fixed number $a \in A$ the sequence of functions $\{f_n\}$ gives rise to a sequence of numbers $\{f_n(a)\}$ by evaluating at a.

Definition 3.1.1

The sequence of functions $\{f_n\}$ is said to *converge pointwise on A* to the function f if for each $a \in A$ the sequence of numbers $\{f_n(a)\}$ converges to $f(a)$. In terms of ϵ we have:

$$\lim_{n \to \infty} f_n = f \quad \text{(pointwise on A)}$$

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if for each $a \in A$ and $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $|f_n(a) - f(a)| < \epsilon$ for all $n > N$.

Example 3.1.2

Let s_n be the nth partial sum of the geometric series $\sum_{n=0}^{\infty} x^n$, i.e $s_n(x) = 1 + x + x^2 + x^3 + ... + x^n$. Then the sequence $\{s_n\}$ converges pointwise to the function given by $f(x) = 1/(1 - x)$ on the set $A = (-1, 1)$.

Example 3.1.3

Let

$$f_n(x) = \begin{cases} -1, & -1 \le x < \frac{-1}{n} \\ nx, & \frac{-1}{n} \le x < \frac{1}{n} \\ 1, & \frac{1}{n} \le x \le 1 \end{cases}$$

and let

$$f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 0, & x = 0 \\ 1, & 0 < x \le 1 \end{cases}.$$

The graph of f_4 is shown in *figure* 3.1.1a and the graph of f is shown in *figure* 3.1.1b. Notice that the functions f_n are continuous while f is not, and yet $\lim f_n = f$ pointwise on $[-1, 1]$. This example shows that pointwise limits do not necessarily behave well with respect to continuity.
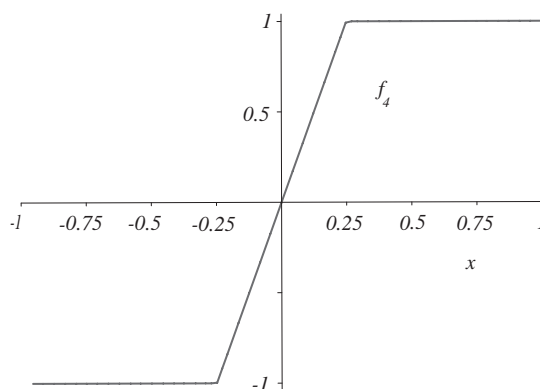
figure 3.1.1a
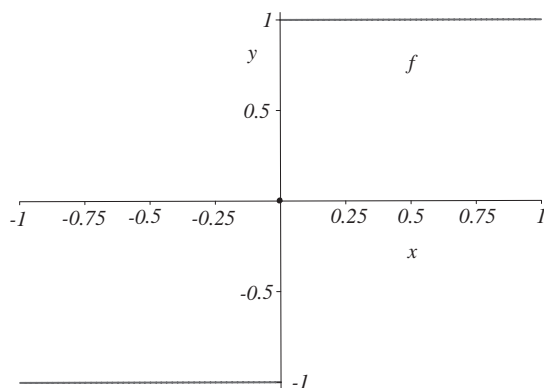
figure 3.1.1b

Example 3.1.4

Let

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x < \frac{1}{2n} \\ 2n - 2n^2x, & \frac{1}{2n} \le x < \frac{1}{n} \\ 0, & \frac{1}{n} \le x \le 1 \end{cases}$$

and let

$$f(x) = 0, \quad 0 \le x \le 1.$$

The graphs of the first three functions of the sequence are shown in *figure 3.1.2*. Notice that $\int_0^1 f_n(x)dx = 1/2$ since the graph of f_n is a triangle with height n and base $1/n$. On the other hand $\int_0^1 f(x)dx = 0$ even though $\lim f_n = f$ pointwise on $[0, 1]$. This shows that pointwise limits do not necessarily behave well with respect to integration.

The above two examples are meant to convince you that pointwise limits do not interact well with the concepts of calculus. The reason for this is that calculus ideas depend on how a function behaves nearby a point, yet the pointwise limit depends only on the values of the functions at each point independently. In this chapter we introduce a stronger form of convergence, *uniform convergence*, which interacts well with the concepts of calculus.
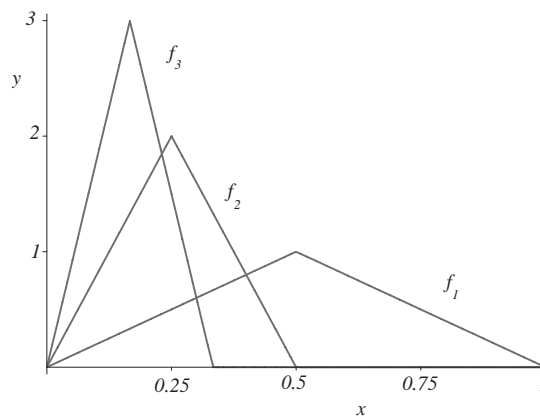
figure 3.1.2

Definition 3.1.5

The sequence of functions $\{f_n\}$ is said to *converge uniformly* to f on A if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|f_n(a) - f(a)| < \epsilon$ for every $n > N$ *and* for every $a \in A$.

Notice that if $\lim f_n = f$ uniformly on A, then also $\lim f_n = f$ pointwise on A. However the sequences in examples 3.1.3 and 3.1.4 converge pointwise but not uniformly on the sets A.

Example 3.1.6

a.) For each $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{n}\sin x$ for $x \in \mathbb{R}$. Then $\lim f_n = 0$ uniformly on $\mathbb{R}$ since if we are given some $\epsilon > 0$ we can choose $N \in \mathbb{N}$ so large that $N > 1/\epsilon$. Then for $n > N$ we have $|f_n(x) - 0| = \frac{1}{n}|\sin(x)| \leq \frac{1}{n} < \frac{1}{N} < \epsilon$ for all $x \in \mathbb{R}$ since $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$.

b.) For each $n \in \mathbb{N}$, let $f_n(x) = xe^{-nx}$ for all $x \geq 0$. The graphs of the first three functions in this sequence are depicted in *figure* 3.1.3. We will show that this sequence of functions converges to zero uniformly on the set $[0, \infty)$. Given some $\epsilon > 0$ we need to find N large enough so that $|f_n(x)| < \epsilon$ for all $x \in [0, \infty)$. To find such an N we will look for the maximum value of $|f_n(x)|$ on the set $[0, \infty)$. Since in this case $f_n(x) \geq 0$ on the set $[0, \infty)$, we can drop the absolute values and look for the maximum value of $f_n(x)$ on
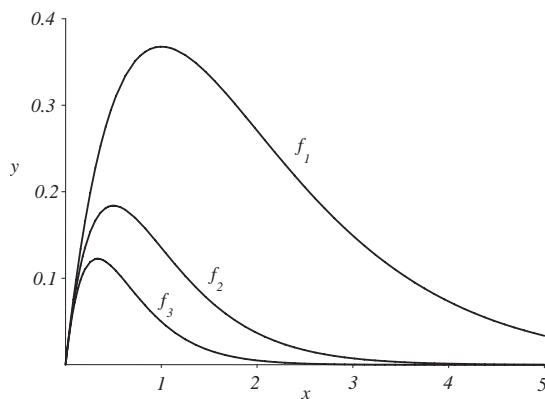
figure 3.1.3

$[0, \infty)$. For this, we use a little calculus. Differentiating, we see that $f_n'(x) = e^{-nx} - nxe^{-nx} = (1-nx)e^{-nx}$, and so the derivative is zero only when $x = 1/n$. It is easy to check that the derivative is positive to the left of $x = 1/n$, and negative to the right of $x = 1/n$, showing that the maximum value of f_n is indeed achieved at $x = 1/n$. Thus the maximum value of f_n on $[0, \infty)$ is given by $f_n(1/n) = \frac{1}{n}e^{-1} = \frac{1}{ne}$. Since this is the maximum value, we know that $f_n(x) \leq \frac{1}{ne}$ for all $x \in [0, \infty)$. Thus if $\epsilon > 0$ is given, we choose $N \in \mathbb{N}$ so that $1/N < e\epsilon$. Then if $n > N$ we have that

$$|f_n(x)| < \frac{1}{ne} < \frac{1}{Ne} < \epsilon,$$

for all $x \in [0, \infty)$, proving that f_n converges to zero uniformly on $[0, \infty)$.

Remark: This method used in this example gives a general method for proving that a sequence $\{f_n\}$ converges uniformly to a function f on a set A. First, note that it is equivalent to show that the sequence $\{f_n - f\}$ converges uniformly to zero on A. To prove this, it suffices to find numbers M_n with $|f_n(x) - f(x)| < M_n$ for all $x \in A$ and $\lim M_n = 0$. Usually, M_n is given by the maximum value of $|f_n - f|$ on A. (If these maxima do not exist, then one needs to consider least upper bounds.)

Of course, now that we understand convergence of sequences of functions, we can also study convergence of series of functions by looking at the sequence of partial sums.

Definition 3.1.7
The series of functions $\sum_{n=1}^{\infty} f_n$ *converges uniformly* to f on A if the sequence of partial sums $s_n = \sum_{j=1}^{n} f_j$ converges uniformly to f on A.

Proposition 3.1.8 (Weierstrass M-test)
Let $\sum f_n$ be a series of functions and M_n a summable sequence of positive real numbers with $|f_n(x)| \leq M_n$ for each $x \in A$. Then $\sum f_n$ converges uniformly on A.

Proof: First, let's notice that the series converges pointwise since if we fix x in A then, since $|f_n(x)| \leq M_n$, the comparison test tells us that $\sum f_n(x)$ converges absolutely. Let $f(x)$ denote the value of $\sum_{n=1}^{\infty} f_n(x)$, for each $x \in A$, so f is a function defined on A and $\sum f_n$ converges to f pointwise on A.

Now we wish to show that this convergence is in fact uniform. Let $\epsilon > 0$ be given. Since $\sum M_n$ converges, we can find $N \in \mathbb{N}$ so that $\sum_{n=N+1}^{\infty} M_n < \epsilon$. Then for all $x \in A$ we have

$$|f(x) - \sum_{n=1}^{N} f_n(x)| = |\sum_{n=N+1}^{\infty} f_n(x)| \qquad (3.1)$$
$$\leq \sum_{n=N+1}^{\infty} |f_n(x)|$$
$$\leq \sum_{n=N+1}^{\infty} M_n$$
$$< \epsilon.$$

Thus $\sum_{n=1}^{\infty} f_n$ converges uniformly to f.
$\square$

As a direct corollary of this and Proposition 2.4.3 we have the following:

Proposition 3.1.9
Let the power series $\sum a_n x^n$ have radius of convergence given by $R > 0$. Then, for any c with $0 < c < R$, this sum converges uniformly on the interval $[-c, c]$.

Proof: From Proposition 2.4.3 we know that $\sum |a_n c^n|$ converges. Let $M_n = |a_n c^n|$ and notice that $|a_n x^n| \leq M_n$ for $x \in [-c, c]$. Applying the M-test we get the desired result.

$\square$

Example 3.1.10

Since we know that the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ on $(-1, 1)$, we can conclude that this convergence is uniform on any subinterval $[-c, c]$, with $0 \le c < 1$.

Remark: Although $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ uniformly on $[-c, c]$ for any c with $0 < c < 1$, it does not converge uniformly on $(-1, 1)$.

We can now prove the three main theorems which demonstrate the compatibility of uniform convergence with the concepts of calculus.

Theorem 3.1.11

Suppose that $\{f_n\}$ is a sequence of continuous functions on the interval $[a, b]$ and suppose that $\{f_n\}$ converges uniformly on $[a, b]$ to a function f. Then f is continuous on $[a, b]$.

Proof: Let $x_0 \in [a, b]$, and $\epsilon > 0$ be given. In order to prove that f is continuous at x_0 we must show how to find $\delta > 0$ so that for all $x \in [a, b]$, with $|x - x_0| < \delta$, we have that $|f(x) - f(x_0)| < \epsilon$.

Now, since we assume that the sequence $\{f_n\}$ converges uniformly to f on $[a, b]$, we can find an $n \in \mathbb{N}$ so that $|f(x) - f_n(x)| < \epsilon/3$, for all $x \in [a, b]$. (In particular, notice that $|f(x_0) - f_n(x_0)| < \epsilon/3$.)

Also, since we are assuming that the function f_n is continuous, we can choose $\delta > 0$ so that for every $x \in [a, b]$ with $|x - x_o| < \delta$, we have that $|f_n(x) - f_n(x_0)| < \epsilon/3$.

Thus, using the old trick of adding zero (twice), we see that if $x \in [a, b]$ with $|x - x_0| < \delta$, we have

$$
\begin{aligned}
|f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\
&\le |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\
&\le \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&\le \epsilon.
\end{aligned}
$$

$\square$

Remark: It is worth noting in this proof that once ϵ is given, we choose a particular f_n which gives the necessary inequality, and then we use the continuity of that particular function. We do not need the full statement of uniform convergence (the "for every $n > N$" part) for this proof.

Theorem 3.1.12

Suppose that $\{f_n\}$ is a sequence of functions that are integrable on the interval $[a, b]$ and suppose that $\{f_n\}$ converges uniformly on $[a, b]$ to a function f which is also integrable on $[a, b]$. Then

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof: Let $c_n = \int_a^b f_n(x)dx$ and let $c = \int_a^b f(x)dx$. Also, let $\epsilon > 0$ be given. We need to show that we can find $N \in \mathbb{N}$ so that $|c_n - c| < \epsilon$, for all $n > N$. Now, since $\{f_n\}$ converges uniformly to f on $[a, b]$, we can choose $N \in \mathbb{N}$ so that $|f_n(x) - f(x)| < \epsilon/(b - a)$ for all $n > N$ and all $x \in [a, b]$. Thus

$$
\begin{aligned}
|c_n - c| &= \left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| \\
&= \left| \int_a^b (f_n(x) - f(x))dx \right| \\
&\leq \int_a^b |f_n(x) - f(x)|dx \\
&\leq \int_a^b \frac{\epsilon}{b - a}dx \\
&\leq \epsilon
\end{aligned}
$$

for all $n > N$. $\square$

Theorem 3.1.13

Suppose that $\{f_n\}$ is a sequence of functions which are differentiable on the interval $[a, b]$. Assume that the derivatives f_n' are integrable and suppose that $\{f_n\}$ converges pointwise to a function f on $[a, b]$. Assume, moreover, that the sequence of derivatives $\{f_n'\}$ converges uniformly on $[a, b]$ to some continuous function g. Then f is differentiable and $f'(x) = g(x)$.

Proof: Let x be an element of $[a, b]$. Since the sequence $\{f_n'\}$ converges uniformly to g on the interval $[a, b]$, it also does on the subinterval $[a, x]$. Also note that since g is continuous on $[a, b]$, it is integrable on that interval. Thus we can apply the above theorem to get

$$\lim_{n \to \infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt.$$

The fundamental theorem of calculus says that $\int_a^x f_n'(t)dt = f_n(x) - f_n(a)$, hence, we have

$$\int_a^x g(t)dt = \lim_{n\to\infty} (f_n(x) - f_n(a)),$$

but the right hand side of this expression is just $f(x) - f(a)$ since $\{f_n\}$ limits to f pointwise. Therefore, we have shown that

$$\int_a^x g(t)dt = f(x) - f(a).$$

Since g is assumed continuous, it also follows from the fundamental theorem of calculus, then, that f is differentiable and that $f'(x) = g(x)$.

$\square$

Remark: A much stronger version of this theorem is true in which the continuity of g is not assumed nor the pointwise convergence of $\{f_n\}$ (except at a single point). The proof is quite a bit trickier, and we don't need the general result for our purposes. The interested student can find the result in *Principles of Mathematical Analysis*, by W. Rudin.

Applying the above results (along with exercise 13) to power series, we get:

Theorem 3.1.14

Suppose that the power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$ has radius of convergence given by $R > 0$. Then the power series $g(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ has radius of convergence equal to R, $f(x)$ is differentiable on $(-R, R)$, and $f'(x) = g(x)$ for $x \in (-R, R)$.

Theorem 3.1.15

Suppose that the power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$ has radius of convergence given by $R > 0$. Then the power series $h(x) = \sum_{n=1}^{\infty} \frac{a_n x^{n+1}}{n+1}$ also has radius of convergence equal to R, $f(x)$ is integrable on $[a, b]$, for any closed interval $[a, b] \subset (-R, R)$, and

$$h(x) = \int_0^x f(s)ds$$

for $x \in (-R, R)$.

Example 3.1.16

a.) Since we know that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $-1 < x < 1$, we can use Theorem 3.1.14 to conclude that $\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ when $-1 < x < 1$.

b.) Applying Theorem 3.1.15 to the same series tells us that $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ converges to $\int_0^x \frac{ds}{1-s} = -\ln(1-x)$ when $-1 < x < 1$. By re-labeling the index, we see that

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

for $-1 < x < 1$. Notice that even though the series converges at $x = -1$, the theorem does not tell us anything about what it converges to. On the other hand, we know from exercise 2.3.12 that the alternating harmonic series converges to $\ln(1/2)$ so the above equality is indeed true on the interval $[-1, 1)$.

EXERCISES 3.1

1. Prove that the sequence of functions in example 3.1.3 does not converge uniformly on $[-1, 1]$.

2. Prove that the sequence of functions in example 3.1.4 does not converge uniformly on $[0, 1]$.

3. Let $f_n(x) = \frac{1}{n} \sin(n^2 x)$. Show that $\{f_n(x)\}$ converges to 0 uniformly on $[-1, 1]$ but that $\lim f_n'(x)$ does not always exist. Explain how this relates to Theorem 3.1.13.

4. For each of the following sequences of functions $\{f_n\}$ defined on the given interval J, determine if the sequence converges pointwise to a limit function f. If the sequence does converge pointwise, determine whether the convergence is uniform.

   a.) $f_n(x) = \frac{x}{x+n}, \quad J = [0, \infty)$.
   b.) $f_n(x) = \frac{nx}{1+n^2 x^2}, \quad J = (-\infty, \infty)$.
   c.) $f_n(x) = \frac{x^n}{1+x^n}, \quad J = [0, \infty)$.
   d.) $f_n(x) = \arctan(nx), \quad J = (-\infty, \infty)$.
   e.) $f_n(x) = e^{-nx}, \quad J = [0, \infty)$.
   f.) $f_n(x) = e^{-nx}, \quad J = [1, \infty)$.
   g.) $f_n(x) = xe^{-nx}, \quad J = (-\infty, \infty)$.

h.) $f_n(x) = x^2 e^{-nx}, \quad J = [0, \infty)$.

5. Assume that f and $\{f_n\}$ are all bounded functions defined on an interval J. Define E_n by $E_n = \sup_{x \in J} |f_n(x) - f(x)|$. Prove that f_n converges to f uniformly on J if and only if $\lim E_n = 0$. (Note that J need not be a closed interval, i.e. it may or may not include one or both of its endpoints or, one or both of the endpoints may be infinite.)

6. Find "closed form" expressions for the following power series and describe the domain on which the series converges to that closed form:

   a.) $\sum_{n=0}^{\infty} n x^n$

   b.) $\sum_{n=0}^{\infty} n^2 x^n$

   c.) $\sum_{n=0}^{\infty} n x^{2n}$

   d.) $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n}$

   e.) $\sum_{n=2}^{\infty} \frac{x^n}{n^2 - n}$

7. Evaluate:

   a.) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

   b.) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

   c.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$

8. (From the William Lowell Putnam Mathematical Competition, 1999)

   Sum the series

   $$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n3^m + m3^n)}.$$

9. a.) Show that $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ for $x \in (-1, 1)$. (Hint: differentiate!)

   b.) Using the trigonometric identity $\frac{\pi}{4} = 4 \arctan(1/5) - \arctan(1/239)$ show that $\pi = 3.14159\dots$

10.   Prove that the series $\sum \frac{\cos nx}{n^2}$ converges uniformly on $A = \mathbb{R}$.

11.   Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on any interval of the form $[\alpha, \infty]$ with α a real number greater than 1. The function defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

is called the *Riemann zeta function.*

12.   a.) Suppose that the power series $f(x) = \sum a_n x^n$ has radius of convergence $R > 0$. Use Proposition 3.1.14 to show that the nth-derivative $f^{(n)}(0)$ is given by $n! a_n$.

b.) Use part a.) to find $f^{(10)}(0)$ for $f(x) = 1/(1 - x)$.

13.   Prove that the two power series $\sum a_n x^n$ and $\sum n a_n x^n$ have the same radii of convergence. (Hint: It is easy to show that if $\sum |n a_n c^n|$ converges then so must $\sum |a_n c^n|$ converge. The other direction needs a comparison argument similar to that in the proof of Proposition 2.4.3.)

3.2   Taylor Series

In exercise 3.1.12 we saw that if $f(x) = \sum_{n=1}^{\infty} a_n x^n$ then the coefficient, a_n, is intimately related to the value of the nth derivative of f at zero, namely

$$f^{(n)}(0) = n! a_n.$$

(Here we should mention the convention that $0! = 1$.) The above observation leads to a method of constructing a power series which might converge to a given function f. In particular, if the series $\sum_{n=1}^{\infty} a_n x^n$ has any chance of converging to $f(x)$, then the coefficients must be given by

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

This leads to the following:

Definition 3.2.1

a.) Let f be an infinitely differentiable function defined on an interval around 0, then the *Maclaurin series* for f is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

b.) Let $a \in \mathbb{R}$, and let f be an infinitely differentiable function defined on an interval containing a, then the *Taylor series for f, centered at a* is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n.$$

Unfortunately, it is not guaranteed that these series actually converge to the original function f. In fact, there are examples of functions which have all the same derivatives at zero, and yet are not equal everywhere (see exercise 14). Still, this observation gives us the only possible candidate for the power series we are looking for, then it is up to us to prove that it really converges to the desired function.

Note: Since the Maclaurin series for f is just a special case of a Taylor series, we will henceforth usually refer to either as a Taylor series, although most of our discussion will be centered on the case $a = 0$.

Example 3.2.2

Consider $f(x) = e^x$. Since $f^{(n)}(x) = e^x$ for all n, we see that $f^{(n)}(0) = 1$ for all n and so the Taylor series for f at 0 (Maclaurin series) is given by $\sum_{n=0}^{\infty} x^n/n!$. We might conjecture, but we have not yet proven that this series converges to $f(x)$.

Given a function f, the nth degree polynomial given by

$$P_n^f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

where $a_n = \frac{f^{(n)}(0)}{n!}$ is called the nth *Taylor polynomial* for f. It is the nth partial sum of the Taylor series we discussed above. (More generally, we can define *Taylor polynomials for f centered at a* as the partial sums of the corresponding Taylor series. We won't bother with them here as this generality just makes the notation more cumbersome and all of the results discussed in this section easily translate to the general case.)

Example 3.2.3

Let $f(x) = \cos(x)$. Then

$$f'(x) = -\sin(x)$$
$$f''(x) = -\cos(x)$$
$$f^{(3)}(x) = \sin(x)$$
$$f^{(4)}(x) = \cos(x)$$

and then this pattern repeats with period 4. Thus we see that $f(0) = 1, f'(0) = 0, f''(0) = -1, f^{(3)}(0) = 0, f^{(4)}(0) = 1$ and this pattern repeats. A good way to write down the general form of the nth derivative at 0 is $f^{(2n+1)}(0) = 0$ and $f^{(2n)}(0) = (-1)^n$. Thus we can write the $2n^{th}$ Taylor polynomial for $\cos(x)$ in the general form

$$P^f_{2n}(x) = \sum_{j=0}^{n}(-1)^j \frac{x^{2j}}{(2j)!}.$$

Since the odd derivatives give zero, it follows that $P^f_{2n+1}(x) = P^f_{2n}(x)$. In particular, we have

$$P^f_0(x) = 1,$$
$$P^f_1(x) = 1,$$
$$P^f_2(x) = 1 - \frac{x^2}{2},$$
$$P^f_3(x) = 1 - \frac{x^2}{2},$$
$$P^f_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!},$$

and so on.

It is sometimes confusing that the sum describing $P^f_{2n}(x)$ only goes up to $j = n$, but notice that the terms are of the form x^{2j} so the highest degree is $2n$, as it should be.

The question of whether or not the Taylor series for f converges to f is the same as the question of whether or not the sequence of Taylor polynomials converges to f. To study this question we begin by looking at how well P^f_n approximates f at $x = 0$. Let's start by looking at P^f_1.

First notice that the equation $y = P_1^f(x)$, which can be rewritten as $y = f(0) + f'(0)x$, describes the tangent line to f at the point $x = 0$. The tangent line is often called the "best linear approximation to f". Let's see why. First of all, recall that the definition of the derivative says that if f is differentiable at 0, then

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}.$$

We can rewrite this as

$$\lim_{x \to 0} \frac{f(x) - f(0) - f'(0)x}{x} = 0$$

i.e.

$$\lim_{x \to 0} \frac{f(x) - P_1^f(x)}{x} = 0.$$

Conversely, assume that f is differentiable at 0 and that some first order polynomial $p(x) = a + bx$ satisfies

$$\lim_{x \to 0} \frac{f(x) - p(x)}{x} = 0. \tag{3.2}$$

Then, multiplying by $\lim_{x \to 0} x = 0$, it follows that

$$\lim_{x \to 0} (f(x) - p(x)) = 0.$$

Since f is differentiable at 0, it must be continuous there, so we can conclude that $f(0) = p(0) = a$. Returning again to (3.2), we see that

$$\lim_{x \to 0} \frac{f(x) - a - bx}{x} = \lim_{x \to 0} \frac{f(x) - f(0) - bx}{x} = 0.$$

Now, since we know that f is differentiable at 0, we know that $\lim_{x \to 0} \frac{f(x) - f(0)}{x}$ exists, hence we can conclude that

$$b = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = f'(0).$$

Thus, we have shown that if f is differentiable at 0, then $P_1^f(x)$ is the only linear polynomial that satisfies (3.2).

Now, we claim that there is a similar result for P_n^f, but before we state this, we introduce some notation:

Definition 3.2.4

Let f and g be two real valued functions on $\mathbb{R}$. We say that f and g *agree to order n at a* if

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x - a)^n} = 0.$$

Now we can state and prove:

Theorem 3.2.5

Let $f : \mathbb{R} \to \mathbb{R}$ be n times differentiable at 0. Then the nth Taylor polynomial, P_n^f is the unique polynomial of degree n which agrees with f to order n at 0.

Proof: First let's prove that P_n^f agrees with f at 0 to order n. To do this, let $g = f - P_{n-1}^f$ and note that:

1.) g is n times differentiable at 0, and hence $g, g', g'', \ldots, g^{(n-1)}$ are all continuous at zero.

2.) $g(0) = g'(0) = g''(0) = \cdots = g^{(n-1)}(0) = 0$.

3.) $g^{(n)}(0) = f^{(n)}(0)$.

Now using L'Hôpital's rule $n - 1$ times, it follows that

$$\lim_{x \to 0} \frac{g(x)}{x^n} = \lim_{x \to 0} \frac{g^{(n-1)}(x)}{n! x} = \frac{1}{n!} \lim_{x \to 0} \frac{g^{(n-1)}(x) - g^{(n-1)}(0)}{x}$$

exists and equals $\dfrac{g^{(n)}(0)}{n!}$. But then

$$\lim_{x \to 0} \frac{f(x) - P_n^f(x)}{x^n} = \lim_{x \to 0} \frac{g(x) - \frac{f^{(n)}(0)}{n!} x^n}{x^n} = \frac{g^{(n)}(0) - f^{(n)}(0)}{n!} = 0.$$

Remark: the only subtlety in the above argument is in making sure that we have applied L'Hôpital's rule appropriately; to fully appreciate this, the reader should think about why it was necessary to single out the three points listed above.

Now let's look at uniqueness. Assume that $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is a polynomial of degree n which agrees with f to order n at 0, i.e.

$$\lim_{x \to 0} \frac{f(x) - p(x)}{x^n} = 0.$$

Using the fact that $\lim_{x\to 0} x^k = 0$ for $k > 0$, we see that

$$\lim_{x\to 0} \frac{f(x) - p(x)}{x^m} = 0$$

for $m = 0, 1, \ldots, n$. In particular, we have that

$$\lim_{x\to 0}(f(x) - p(x)) = 0.$$

By continuity, we conclude that $f(0) = p(0) = a_0$. Next, apply L'Hôpital's rule to

$$\lim_{x\to 0} \frac{f(x) - p(x)}{x} = 0$$

to conclude that $f'(0) = p'(0) = a_1$. This result allows us to apply L'Hôpital's rule twice to

$$\lim_{x\to 0} \frac{f(x) - p(x)}{x^2} = 0$$

to conclude $f''(0) = p''(0) = 2a_2$. Continuing in this fashion (by induction if you insist on being pedantic), we can conclude $f^{(m)}(0) = p^{(m)}(0) = m!a_m$ for $m = 1, 2, \ldots, n - 1$. For the final coefficient, a_n, L'Hôpital's rule isn't adequate since we aren't assuming continuity of f^n at 0. Here we apply L'Hôpital's rule $n - 1$ times to get

$$\begin{aligned}
\lim_{x\to 0} \frac{f(x) - p(x)}{x^n} &= \lim_{x\to 0} \frac{f^{(n-1)}(x) - n!a_n x - (n-1)!a_{n-1}}{n!x} \\
&= \lim_{x\to 0} \frac{f^{(n-1)}(x) - f^{(n-1)}(0) - n!a_n x}{xn!} \\
&= \lim_{x\to 0} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{xn!} - n!a_n \\
&= 0.
\end{aligned}$$

from which we conclude that $a_n = f^{(n)}(0)/n!$. Putting all of this together, we find $p = P_n^f$ $\square$

The uniqueness of P_n^f in the above theorem is often useful in determining Taylor polynomials. To prove that a given n^{th} degree polynomial is the Taylor polynomial for f, we simply need to show that it agrees with f to order n at 0. To make use of this idea, we need to introduce a little notation:

If $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is a polynomial, and $0 \le k \le n$ we define $[P]_k$ to be the degree k polynomial given by truncating P after the x^k term, i.e.

$$[P]_k(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k.$$

For example, if $P(x) = 3 - x + 4x^2 + 7x^4 - 4x^5$, then $[P]_4(x) = 3 - x + 4x^2 + 7x^4$, while $[P]_3(x) = [P]_2(x) = 3 - x + 4x^2$. Notice that if P and Q are two polynomials, then $[P + Q]_k = [P]_k + [Q]_k$.

Using the uniqueness from the above theorem, it is not too hard to prove the following:

Theorem 3.2.6

Let f and g be real valued functions defined on a neighborhood of 0 which have n derivatives at 0. Then

1.) $P_n^{f+g} = P_n^f + P_n^g$.

2.) $P_n^{fg} = [P_n^f P_n^g]_n$.

3.) and if $g(0) = 0$, then $P_n^{f \circ g} = [P_n^f \circ P_n^g]_n$.

4.) $P_{n-1}^{f'} = (P_n^f)'$.

5.) Let $h(x) = \int_0^x f(t)\, dt$, then $P_n^h(x) = \int_0^x P_{n-1}^f(t)\, dt$.

Remark: As a special case of item 3.), one also has

4.) if $g(0) = 0$, then $P_n^{1/(1-g)} = [1 + P_n^g + (P_n^g)^2 + \cdots + (P_n^g)^n]_n$.

Example 3.2.7

We know that the fifth degree Taylor polynomial for $\cos(x)$ is

$$P_5^{\cos}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

and the fifth degree Taylor polynomial for $\sin(x)$ is

$$P_5^{\sin}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

so from the above theorem, we can conclude that the fifth degree Taylor polynomial for $\sin(x) + \cos(x)$ is

$$P_5^{\cos(x)+\sin(x)} = 1 + x - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

while the fifth degree Taylor polynomial for the product, $\cos(x)\sin(x)$, is given by

$$
\begin{aligned}
P_5^{\cos(x)\sin(x)}(x) &= \left[(1 - \frac{x^2}{2} + \frac{x^4}{4!})(x - \frac{x^3}{3!} + \frac{x^5}{5!})\right]_5 \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^3}{2} + \frac{x^5}{2 \cdot 3!} + \frac{x^5}{4!} \\
&= x - \frac{4x^3}{3!} + \frac{16x^5}{5!}.
\end{aligned}
$$

Remark: On the other hand, using the composition rule, we can see that the fifth degree Taylor polynomial for $\sin(2x)/2$ is given by

$$
\begin{aligned}
P_5^{\sin(2x)/2} &= \frac{1}{2}\left((2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}\right) \\
&= x - \frac{4x^3}{3!} + \frac{16x^5}{5!},
\end{aligned}
$$

the same as above.

Example 3.2.8

Setting $h(x) = e^{1-\cos(x)}$ we have

$$
\begin{aligned}
P_5^h(x) &= \left[1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{24}\right)^2 + \frac{1}{6}\left(\frac{x^2}{2} - \frac{x^4}{24}\right)^3\right]_5 \quad (3.3) \\
&\quad + \left[\frac{1}{24}\left(\frac{x^2}{2} - \frac{x^4}{24}\right)^4 + \frac{1}{120}\left(\frac{x^2}{2} - \frac{x^4}{24}\right)^5\right]_5 \\
&= \left[1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{24}\right)^2\right]_5 \\
&= 1 + \frac{x^2}{2} + \frac{x^4}{12},
\end{aligned}
$$

and if $k(x) = \int_0^x h(t)\, dt$, we have

$$
P_5^k(x) = \int_0^x P_4^h(t)\, dt = x + \frac{x^3}{6} + \frac{x^5}{60}.
$$

Notice that we have used Theorem 3.2.5 to compute Taylor polynomials of complicated functions without explicitly computing their derivatives. Thus, we can now use the fundamental relation $a_n = f^{(n)}(0)/n!$ to compute values of $f^{(n)}(0) = n!a_n$. In particular, for $h(x)$ in the above example we have that $a_0 = 1, a_1 = 0, a_2 = 1/2, a_3 = 0, a_4 = 1/12$, and $a_5 = 0$, so we see that $h'(0) = 0, h''(0) = 1, h^{(3)}(0) = 0, h^{(4)}(0) = 2$, and $h^{(5)}(0) = 0$. The reader may find it amusing to check these values by explicitly computing the fifth derivative of $h(x)$.

Theorem 3.2.5 tells us how well P_n^f approximates f at the point $x = 0$. To understand whether the sequence of Taylor polynomials $\{P_n^f\}$ converges to f in a neighborhood of zero we need more explicit control on the differences $|f(x) - P_n^f(x)|$. The next theorem gives us that control.

Theorem 3.2.9 (Taylor's Theorem)
Suppose that $f, f', f'', ..., f^{(n+1)}$ are all defined on the interval $[0, x]$. Let $R_n^f(x)$ be defined by

$$R_n^f(x) = f(x) - P_n^f(x)$$
$$= f(x) - \left(f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n \right).$$

Then

$$R_n^f(x) = \frac{f^{(n+1)}(t)}{(n+1)!}x^{n+1}, \quad \text{for some } t \in (0, x).$$

Example 3.2.10
The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly to e^x on any closed interval $[a, b]$. To prove this, first notice that the radius of convergence for this series is infinite, so we know the series converges uniformly on any closed interval. In particular, we know that for any given x,

$$\lim_{n \to \infty} x^n/n! = 0.$$

Now we need only show that the series converges to e^x pointwise. Fix some $x \in \mathbb{R}$ and let $P_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$, then Taylor's Theorem tells us that

$$|e^x - P_n(x)| = e^t |x|^{n+1}/(n+1)!$$

for some t between 0 and x. But then $e^t < e^{|x|}$ so we get

$$|e^x - P_n(x)| \leq \frac{e^{|x|}|x|^{n+1}}{(n+1)!}.$$

We have remarked above that the sequence $|x|^n/n!$ converges to zero as n goes to infinity, so, given $\epsilon > 0$, we can choose N so large that $|x|^{n+1}/n + 1! < e^{-|x|}\epsilon$ for all $n > N$. Then, for all $n > N$, we have that $|e^x - P_n(x)| < \epsilon$. Thus $P_n(x)$ converges to e^x.

Another valuable application of Taylor's Theorem is that it gives us estimates on how well a Taylor polynomial approximates the original function.

Example 3.2.11

As above, the nth Taylor polynomial for $f(x) = e^x$ is given by $P_n^f(x) = \sum_{j=0}^n x^j/j!$ and Taylor's Theorem gives us an estimate for the error term $|f(x) - P_n^f(x)|$. Let's use this to find a decimal approximation for $f(1) = e$ which is correct to 4 decimal places. To do this, we must figure out which n will give us a small enough error. By Taylor's Theorem, we have

$$|e - P_n^f(1)| \leq \left| \frac{f^{(n+1)}(t)}{(n+1)!}(1)^n \right| = \left| \frac{f^{(n+1)}(t)}{(n+1)!} \right|$$

for some t with $0 < t < 1$. Now, as above, $f^{(n)}(x) = e^x$ so, if $0 < t < 1$, then $f^{(n+1)}(t) < e$. Thus we can estimate that

$$\left| \frac{f^{(n+1)}(t)}{(n+1)!} \right| \leq \frac{e}{(n+1)!}.$$

Well now we have to cheat a little bit and say, "everyone knows that $e < 3$" (a proof of this fact is not too hard, but it requires revisiting the definition of e^x.) Anyway, using this fact, we see that we can make the error estimate less than 10^{-5} as long as $3/(n+1)! < 10^{-5}$, i.e. we need $(n+1)! > 3 \times 10^5$. Making a list of values of $n!$ we have

$$1! = 1$$
$$2! = 2$$
$$3! = 6$$
$$4! = 24$$
$$5! = 120$$
$$6! = 720$$
$$7! = 5,040$$
$$8! = 40,320$$
$$9! = 362,880$$
$$10! = 3,628,800.$$

Since $9! > 3 \times 10^5$ we can take $P_8^f(1)$ to approximate $e = f(1)$ and be sure that it is correct to four decimal places. I.e.

$$e - (1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320}) < 10^{-5}.$$

Evaluating this sum on a calculator (what a cheat!) yeilds $e \approx 2.71827$.

EXERCISES 3.2

1. Find the Maclaurin series for the following functions.

    (a) $f(x) = \dfrac{1}{1 - x}$ (b) $g(x) = \dfrac{1}{1 + x^2}$

    (c) $F(x) = \sin x$ (d) $G(x) = e^{x^2}$

2. Find the Taylor series centred at a for $f(x) = 1 + x + x^2 + x^3$ when

    (a) $a = 0$ (b) $a = 1$ (c) $a = 2$

3. Prove that the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converges uniformly to $\cos x$ on any closed interval $[a, b]$.

4. Find the sum of these series

    (a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$ (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{6^{2n}(2n)!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n(n+1)!}$

5. Use the 5th degree Taylor polynomial for $f(x) = e^x$ and Taylor's theorem to obtain the estimate
    $$\frac{1957}{720} \le e \le \frac{1956}{719}$$

6. For what values of x do the following polynomials approximate $\sin x$ to within 0.01

    (a) $P_1(x) = x$ (b) $P_3(x) = x - x^3/6$ (c) $P_5(x) = x - x^3/6 + x^5/120$

7. How accurately does $1 + x + x^2/2$ approximate e^x for $x \in [-1, 1]$? Can you find a polynomial that approximates e^x to within 0.001 on this interval?

8. Use Taylor's Theorem to approximate $\cos(\pi/4)$ to 5 decimal places.

9. Find the Taylor series for $x^2 \sin(x^2)$.

10. Find the 10th degree Taylor polynomial for the following functions:

    a.) $\cos(x^2)$

    b.) $\sin(2x)$

    c.) e^{x+1}

    d.) $e^{x^2} \cos(x^3)$

    e.) $\frac{\sin(x^2)}{1+x^2}$

11. a.) Find the Taylor series for $f(x) = \frac{1-\cos(x^6)}{x^{12}}$.

    b.) Use part a.) to determine $f^{(n)}(0)$ for $n = 0, 1, 2, 3, ..., 12$.

12. Evaluate

$$\int_0^1 e^{-x^2} \, dx$$

to within 0.001.

13. Approximate $\int_0^1 \frac{\sin(x)}{x} \, dx$ to 3 decimal places.

14. Consider the function

$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x^2}, & x > 0 \end{cases}.$$

a.) Show that f is continuous at 0.

b.) Show that f is differentiable at 0 and that $f'(0) = 0$. (Hint: you must use the definition of the derivative to prove this.)

c.) Show that $f'(x)$ is differentiable at 0 and that $f''(0) = 0$.

d.) Show by induction that $f^{(n)}(x)$ is differentiable at 0 and that $f^{(n+1)}(0) = 0$, for all $n \in \mathbb{N}$.

e.) To what function does the Taylor series for f converge?

15. Let r be a nonzero real number and define the generalized binomial coefficient $\binom{r}{k}$ by $\binom{r}{0} = 1$, and

$$\binom{r}{k} = \frac{r \cdot (r-1) \cdot (r-2) \cdot \cdots \cdot (r-k+1)}{k!}, \quad \text{for} \quad k \geq 1.$$

a.) Prove the formula

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

for $-1 < x < 1$.

b.) Write out the first 5 terms in this series for $r = -1, r = 1/2$, and $r = 3$.

16. (Error term for alternating series) Suppose $\{b_n\}$ is a decreasing sequence of positive numbers with $\lim b_n = 0$. From the alternating series test we know $\sum_{n=1}^{\infty} (-1)^n b_n$ converges, say to s. Let s_n denote the nth partial sum of the alternating series. Show that $|s - s_n| < b_{n+1}$ for all n.

17. Use the method of generating functions (see problem 7) to discover the formula $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ as follows:

First, let $a_n = 1+2+3+\cdots+n$ for all $n \in \mathbb{N}$ and note that $a_n = a_{n-1}+n$, and $a_1 = 1$. Next, define a generating function $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Now insert the recursive relation for a_n into the definition of $f(x)$ to get

$$f(x) = xf(x) + \frac{x}{(1-x)^2}.$$

From this conclude that

$$f(x) = \frac{x}{(1-x)^3}.$$

Finally, twice differentiating $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ yields

$$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1)x^{n-2},$$

which when substituted into the expression for $f(x)$ yields $a_n = \frac{n(n+1)}{2}$.

3.3 Complex Numbers

To introduce *complex numbers* we define a new "number", usually denoted by i which satisfies $i^2 = -1$. That is, i is a "square root of -1". Notice that $-i$ is also a square root of -1, so be careful not to mistake i for *the* square root of -1.

Once we accept the use of this new number, we can define the set of *complex numbers* as

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}.$$

If $z = a + ib$ is a complex number, we call a the *real part* of z and b the *imaginary part* of z. These are denoted respectively by $\text{Re}(z)$ and $\text{Im}(z)$ Two complex numbers z_1 and z_2 are equal when their real and imaginary parts are equal, i.e. $z_1 = z_2$ if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1)$ $=\text{Im}(z_2)$.

Complex numbers can be added and multiplied using the following definition.

Definition 3.3.1

Let $z, w \in \mathbb{C}$ with $z = a + ib$, and $w = c + id$, $a, b, c, d \in \mathbb{R}$. Then
a.) $z + w = (a + c) + i(b + d)$
b.) $zw = (ac - bd) + i(ad + bc)$.

One thing to be careful about is that imaginary numbers can't be made into an "ordered field" (see Appendix A), so it makes no sense to use inequalities with complex numbers. On the other hand, there is a real quantity attached to a complex number called the modulus that behaves like the absolute value for real numbers, in particular the triangle inequality holds.

Definition 3.3.2

Let $z = a + ib$ with $a, b \in \mathbb{R}$. The *modulus* of z is given by $\sqrt{a^2 + b^2}$. The modulus of z is denoted by $|z|$.

Proposition 3.3.3

Let $z, w \in \mathbb{C}$. Then
a.) $|\text{Re}(z)| \leq |z|$
b.) $|\text{Im}(z)| \leq |z|$
c.) $|z + w| \leq |z| + |w|$.

It is useful to think of the complex numbers geometrically in terms of the *complex plane*. Any complex number $z = a + ib$ can be identified to

the ordered pair of real numbers (a, b), which is understood geometrically as a point in the Cartesian plane. The modulus of z then has the geometric interpretation as the distance between the point (a, b) and the origin $(0, 0)$. A little thought reveals then that if z and w are complex numbers then $|z - w|$ is the distance between the points in the plane related to z and w. In particular, if $\epsilon > 0$ and $c \in \mathbb{C}$ the set $\{z \in \mathbb{C} \mid |z - c| < \epsilon\}$ is the open disc of points centered at c with radius ϵ.

Much of the work on sequences and series of real numbers carries over directly to sequences and series of complex numbers if we just use the complex modulus in place of the absolute values. In particular if we let $\{c_n\}$ be a sequence of complex numbers, we have the following

Definition 3.3.4

The sequence $\{c_n\}$ of complex numbers converges to the complex number c if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|c_n - c| < \epsilon$$

for every $n > N$. As usual, this is written as $\lim_{n \to \infty} c_n = c$.

In general terms, we have $\lim_{n \to \infty} c_n = c$ if "eventually" all of the terms of the sequence are situated in the open disc of radius ϵ centered at c.

Proposition 3.3.5

Let $\{c_n\}$ be a sequence of complex numbers with $c_n = a_n + ib_n$, $(a_n, b_n \in \mathbb{R})$. Let $c = a + ib$ be a complex number. Then we have $\lim_{n \to \infty} c_n = c$ if and only if $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$.

Proof: This follows from the inequalities

$$|a_n - a| \leq |c_n - c|, \ |b_n - b| \leq |c_n - c|, \ \text{and} \ |c_n - c| \leq |a_n - a| + |b_n - b|,$$

which all follow from Proposition 3.3.3.

□

Most of the results of chapter 2 (especially those in section 2.3) can be generalized to complex sequences. However, at the moment we are more interested in complex series. Let $\{c_n\}$ be a sequence of complex numbers. We can form a *sequence of partial sums*

$$s_n = \sum_{j=0}^{n} c_j.$$

We say that the series $\sum c_n$ *converges* if the sequence $\{s_n\}$ converges. Otherwise, it *diverges*. A direct consequence of Proposition 3.3.5 is the following:

Proposition 3.3.6
   Let $\{c_n\}$ be a sequence of complex numbers with $c_n = a_n + ib_n$ $(a_n, b_n \in \mathbb{R})$. Then $\sum c_n$ converges if and only if both $\sum a_n$ and $\sum b_n$ converge. Furthermore, if $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = b$ then $\sum_{n=0}^{\infty} c_n = a + ib$.

Proof: This follows from applying Proposition 3.3.5 to the partial sums of the series.

$\square$

   The above Proposition is not much help in determining the convergence of geometric series like

$$\sum_{n=0}^{\infty} \left(\frac{1+i}{2} \right)^n$$

since it is not that easy to find the real and imaginary parts of $\left(\frac{1+i}{2} \right)^n$. On the other hand, this Proposition does help to prove for complex series many of the results we know about real series, for example, we have

Proposition 3.3.7
   Suppose that $\{c_n\}$ is a sequence of complex numbers and that $\sum c_n$ converges. Then $\lim_{n \to \infty} c_n = 0$.

Proof: Let $c_n = a_n + ib_n$ where a_n and b_n are real. Then since $\sum c_n$ converges, we know that both $\sum a_n$ and $\sum b_n$ converge. These are real series, so we know from Proposition 2.1.7 that $\lim a_n = 0$ and $\lim b_n = 0$. Finally, from Proposition 3.3.5 we conclude that $\lim c_n = 0$.

$\square$

   To help address the question of convergence of the above series we introduce

Definition 3.3.8
   We say that the series $\sum c_n$ *converges absolutely* if the series $\sum |c_n|$ converges.

   Notice that the series $\sum |c_n|$ has real, nonnegative terms so we can check its convergence using the tests from section 2.2.

Proposition 3.3.9
   If a series converges absolutely, then it converges.

Proof: Let $c_n = a_n + ib_n$ where a_n and b_n are real. Then since $|a_n| \le |c_n|$, the convergence of $\sum |c_n|$ implies the convergence of $\sum |a_n|$ (by comparison). Likewise, since $|b_n| \le |c_n|$, the convergence of $\sum |c_n|$ also implies the convergence of $\sum |b_n|$. Now, since $\sum a_n$ and $\sum b_n$ are real series, we can apply proposition 2.3.1 to conclude that they both converge. Then, by Proposition 3.3.6, we see that $\sum c_n$ converges.

$\square$

Example 3.3.10

The terms of the series $\sum_{n=0}^{\infty} \left(\frac{1+i}{2} \right)^n$ have modulus

$$\left| \left(\frac{1+i}{2} \right)^n \right| = \left(\frac{\sqrt{2}}{2} \right)^n.$$

Since that sequence is summable, the series is absolutely convergent.

Finally, for power series, we have the following:

Proposition 3.3.11

Let $\sum a_n z^n$ be a complex power series and assume that for some fixed $c \in \mathbb{C}$ we know that $\sum a_n c^n$ converges. Then $\sum a_n z^n$ converges for any z with $|z| < |c|$.

Proof: The proof is similar to that of Proposition 2.4.3. We begin by noting that since $\sum a_n c^n$ converges, we know that $\lim a_n c^n = 0$ and thus we can conclude that the sequence $\{a_n c^n\}$ is bounded in the sense that there is some real number $M > 0$ so that $|a_n c^n| < M$ for all $n \in \mathbb{N}$. (This follows from 3.3.5 and the analogous fact for real sequences.) Now choose z with $|z| < |c|$ and let $d = |z|/|c|$. Then

$$\begin{aligned}
\sum |a_n z^n| &= \sum |a_n||z|^n \\
&= \sum |a_n||c|^n d^n \\
&= \sum |a_n c^n| d^n \\
&\le \sum M d^n.
\end{aligned} \tag{3.4}$$

But the last of these sums is finite since $d < 1$ so we conclude by the comparison test that the first sum is also finite, i.e. $\sum a_n z^n$ converges absolutely.

$\square$

Corollary 3.3.12

Let $\sum a_n z^n$ be a complex power series. Then either

1.) $\sum a_n z^n$ converges for all $z \in \mathbb{C}$, or

2.) there is a nonnegative real number R so that $\sum a_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.

In case 2.), R is called the *radius of convergence*, in case 1.) the radius of convergence is said to be infinite. In the second case, there is no information about convergence on the circle $|z| = R$. Convergence at such points must be dealt with in more detail in order to describe the full domain of convergence.

Example 3.3.13

Thinking of power series in the complex domain gives some intuition about "why" the Taylor series, $\sum_{n=0}^{\infty} (-1)^n x^{2n}$, for $f(x) = 1/(1 + x^2)$, converges only for $|x| < 1$ even though f makes perfectly good sense for all real values of x. The point is that if we think of x as a complex variable, then f has a "singularity" at $x = i$ and $x = -i$, so the radius of convergence for $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ cannot possibly be bigger than 1.

Example 3.3.14

The power series $\sum_{n=0}^{\infty} x^n/n!$ converges to $f(x) = e^x$ for all real x. Hence its radius of convergence is infinite. Therefore the complex power series $\sum_{n=0}^{\infty} z^n/n!$ converges for all complex z. It makes sense to think of the function defined by this complex power series to be an *extension* of $f(x) = e^x$. That is, we use the power series $\sum_{n=0}^{\infty} z^n/n!$ to *define* e^z for complex numbers z. Of course, just naming this e^z does not mean that it necessarily behaves like an exponential. For instance, it is not at all clear from this definition that the usual exponent rule, $e^{z+w} = e^z e^w$ is still valid. It turns out that this rule does remain valid for complex exponents, but we won't go into the general proof of that now. Instead, let's just look at the special case where z and w are purely imaginary, i.e. $z = ix$ and $w = iy$, x and y real. Before looking at the rules of exponents, we need to look a little more closely at the power series for e^z when $z = ix$. Noticing that $z^2 = -x^2, z^3 = -ix^3, z^4 = x^4$

and this pattern repeats with period four, we get that

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

$$= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \dots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + \dots$$

$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots).$$

Recalling, however, that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

and that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

we see that

$$e^{ix} = \cos(x) + i\sin(x).$$

Now it is just a matter of using some addition of angle formulas to see that $e^{ix+iy} = e^{ix}e^{iy}$. Indeed

$$\begin{aligned} e^{ix+iy} &= \cos(x+y) + i\sin(x+y) \\ &= (\cos(x)\cos(y) - \sin(x)\sin(y)) + i(\cos(x)\sin(y) + \cos(y)\sin(x)) \\ &= (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) \\ &= e^{ix}e^{iy}. \end{aligned}$$

(Actually, this author admits to having a hard time remembering the addition of angle formulas, but since he knows that $e^{ix} = \cos(x) + i\sin(x)$, and the exponent laws, he can always "rederive" the angle addition formulas.) By the way, the formula $e^{ix} = \cos(x) + i\sin(x)$ also gives some interesting identities by plugging in particular values of x, for example

$$e^{i\pi} = -1.$$

EXERCISES 3.3

1. Find the domains of convergence of the following complex power series

   a.) $\sum \frac{z^n}{n^2}$

   b.) $\sum \frac{z^n}{2^n n^2}$

   c.) $\sum \frac{z^n}{n!}$

   d.) $\sum (-1)^n \frac{z^n}{n! 2^n}$

   e.) $\sum z^n$.

2. Show that if $|z| < 1$ then $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

3. Using exercise 2, evaluate $\sum_{n=0}^{\infty} \left(\frac{1+i}{2} \right)^n$.

4. Evaluate $\sum_{n=0}^{\infty} n \left(\frac{1+i}{2} \right)^n$.

5.   a.) Show that any complex number $z = a + ib$ can be written in the *polar form*, $z = r(\cos(\theta) + i \sin(\theta))$, where $r = |z|$ and θ is the angle between the real axis and the line segment passing from the origin to the point (a, b).

   b.) Notice that the polar form of z can be rewritten using the complex exponential, $z = r(\cos(\theta) + i \sin(\theta)) = re^{i\theta}$. Use this to prove *deMoivre's theorem*:

   $$z^n = r^n(\cos(n\theta) + i \sin(n\theta)).$$

   c.) Interpret deMoivre's theorem geometrically in the complex plane.

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