MATH 4000/6000 – Learning objectives to meet for Exam #3

The exam will be over §§4.1–4.2; this includes all material covered since your previous exam through the lecture on Monday, 4/15.

What to be able to state

Basic definitions

You should be able to give precise descriptions of all of the following:

- homomorphism
- kernel of a homomorphism
- ideal of a commutative ring
- principal ideal
- the notation $\langle x \rangle$ and (more generally) $\langle x_1, \ldots, x_n \rangle$
- definition of the quotient ring R/I (including what the elements are and how the operations are defined)
- prime ideal
- isomorphism of rings
- direct product of rings
- $f(x)$ splits over F

Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class or on homework, describe the components and main ideas of the proof.

- the kernel of a homomorphism is an ideal
- every ideal of $\mathbf{Z}$ is principal
- every ideal of $F[x]$ (with F a field) is principal
- If $\phi\colon R \to S$ is a homomorphism, then ϕ is one-to-one if and only if $\ker(\phi) = \langle 0 \rangle$.
- Fundamental Homomorphism Theorem, as stated in class (remember that our statement is more general than the one in Shifrin's book!)
- R/I is a domain if and only if I is a prime ideal
- If $f(x) \in F[x]$ has degree $n \geq 1$, then $F[x]/\langle f(x) \rangle = \{\overline{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}} : \text{all } a_i \in F\}$. Moreover, each element of $F[x]/\langle f(x) \rangle$ has a unique expression as $\overline{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}}$.

- If $p(x) \in F[x]$ is irreducible, then $K = F[x]/\langle p(x) \rangle$ is a field.
- If $f(x) \in F[x]$ is reducible, then $K = F[x]/\langle f(x) \rangle$ has zero divisors.
- If $p(x) \in F[x]$ is irreducible, then $p(x)$ has the root $\alpha = \bar{y}$ in $K = F[y]/\langle p(y) \rangle$.
- If $f(x) \in F[x]$ is any nonconstant polynomial, there is a field extension K of F in which $f(x)$ has a root.
- If $f(x) \in F[x]$ is any nonconstant polynomial, there is a field extension K of F over which $f(x)$ splits.

What to be able to do

You are expected to know how to use the methods described in class/developed on HW to solve the following problems (not comprehensive!).

- Establish properties of quotient rings R/I by relating these back to the properties of the original ring R.
- Perform computations in quotient rings R/I (for example, multiplying two elements $\mathbf{Q}[x]/\langle x^3 + x \rangle$).
- Find inverses of elements in $F[x]/\langle f(x) \rangle$, via the Euclidean algorithm and back substitution.
- Establish isomorphisms between rings using the Fundamental Homomorphism Theorem.

Practice problems

1. Let R be a commutative ring.

   (a) What does it mean to say that a subset I of R is an ideal of R?

   (b) If I and J are ideals of R, prove that $I \cap J$ is also an ideal of R.

   (c) Suppose now that $R = \mathbf{Z}$. If $I = \langle 6 \rangle$ and $J = \langle 10 \rangle$, which ideal of $\mathbf{Z}$ is $I \cap J$? Express your answer in the form $\langle n \rangle$ for a positive integer n. **No justification required.**

2. Let R and S be commutative rings.

   (a) What does it mean to say that $\phi\colon R \to S$ is a homomorphism?

   (b) If $\phi\colon R \to S$ is a homomorphism, what do we mean by the kernel $\ker(\phi)$?

   (c) Suppose that $\phi\colon R \to S$ is a homomorphism. Show that if S is an integral domain, then $\ker(\phi)$ is a prime ideal of R.

3. (a) If R is a commutative ring and I is an ideal of R, what do we mean by the quotient ring R/I? Describe both the elements of R/I and the definition of addition and multiplication in R/I.

   (b) Since $\mathbf{Z}_{17}[t]$ is a domain, there are no solutions to $x^2 = 0$ in $\mathbf{Z}_{17}[t]$ other than $x = 0$. What are all solutions to $x^2 = 0$ in the ring $\mathbf{Z}_{17}[t]/\langle t^3 \rangle$? How many are there? Justify your answer.

4. (a) State the Fundamental Homomorphism Theorem, as given in class.

   (b) Is $\mathbf{Z}_7[x]/\langle x^2 + 1 \rangle$ isomorphic to $\mathbf{Z}_7 \times \mathbf{Z}_7$? If the answer is no, explain why not. If yes, prove there is an isomorphism by using the Fundamental Homomorphism Theorem.

   (c) For this problem assume as known that the real number π is not algebraic over $\mathbf{Q}$.

   Is $\mathbf{Q}[x] \cong \mathbf{Q}[\pi]$? If the answer is no, explain why not. If yes, prove there is an isomorphism by using the Fundamental Homomorphism Theorem.

   (d) Is there a real number α for which $\mathbf{Q}[\alpha] \cong \mathbf{Q}[x]/\langle x^2 \rangle$? Justify your answer.

5. Let R and S be rings. Assume neither R nor S is the zero ring. Is it possible for $R \times S$ to be an integral domain? If so, give an example of R and S where this happens. If not, prove this is impossible.

6. Let $K = \mathbf{Z}_3[t]/\langle t^3 + t + 1 \rangle$. Let $\alpha = \bar{t} \in K$.

   (a) Is K a domain? Is K a field?

   (b) Find α^{-1} in K. Express your answer in the form $c_0 + c_1\alpha + c_2\alpha^2$, where $c_0, c_1, c_2 \in \mathbf{Z}_3$.

   (c) With $f(x) = x^3 + x + 1$, find all of $f(\alpha)$, $f(\alpha^2)$ and $f(\alpha^3)$. Express your answers in the form $c_0 + c_1\alpha + c_2\alpha^2$, where $c_0, c_1, c_2 \in \mathbf{Z}_3$.