MATH 4000/6000 – Homework #2

posted January 24, 2024; due by end of day on January 31, 2024

Mathematics is not a deductive science – that's a cliché. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork.
— Paul Halmos (1916–2006)

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

0. [For practice only: **not to turn in!**] Prove the *law of cancelation* in $\mathbb{Z}$: If $ab = ac$ and $a \neq 0$, then $b = c$. *Hint:* If $ab = ac$, then $a(b - c) = 0$. Now use a result from HW #1.

1. For each pair of integers a, b, define the set

$$\mathrm{CD}(a,b) = \{d \in \mathbb{Z} : d \mid a \text{ and } d \mid b\}.$$

Suppose a, b, q, r are integers with $a = bq + r$. Prove that $\mathrm{CD}(a,b) = \mathrm{CD}(b,r)$.

Remark. As discussed in class, it is this result that justifies the Euclidean algorithm as a method of computing gcds. Namely, if we apply this result repeatedly as we step through the Euclidean algorithm, we eventually find that $CD(a,b) = CD(0,r)$, where r is the last nonzero remainder. Hence, the set of common divisors of a, b is the same as the set of divisors of r.

2. Let a, b be integers, not both 0. We showed in class every common divisor of a and b divides $\gcd(a,b)$. Hence, the number $d = \gcd(a,b)$ is a positive integer with the following property:

$$d \text{ divides } a \text{ and } b\text{, and every common divisor of } a \text{ and } b \text{ divides } d. \tag{$\dagger$}$$

Prove that $\gcd(a,b)$ is the *only* positive integer d that satisfies ($\dagger$).

Remark. This exercise shows that ($\dagger$) could have been taken as the **definition** of $\gcd(a,b)$. That's the approach taken in Shifrin's textbook, for instance.

3. (a) Prove that if $a \mid x$ and $b \mid y$, then $ab \mid xy$.
   (b) Prove that if $d = \gcd(a,b)$, then $\gcd(a/d, b/d) = 1$.
   (c) Prove or give a counterexample: If $d = \gcd(a,b)$, then $\gcd(a/d, b) = 1$.

4. Suppose a, b, and n are positive integers for which $\gcd(a,n) = \gcd(b,n) = 1$. Prove or give a counterexample: $\gcd(ab, n) = 1$.

Hint: One approach starts by proving the following lemma: $\gcd(A,B) > 1$ if and only if there is a common prime p dividing both A and B.

5. Let p be a prime number. Prove that if $a^2 \equiv b^2 \pmod{p}$, then $a \equiv b \pmod{p}$ or $a \equiv -b \pmod{p}$.

6. (Divisibility in Pythagorean triples) Recall that an ordered triple of integers x, y, z is called **Pythagorean** if $x^2 + y^2 = z^2$.
   (a) Show that in any Pythagorean triple, at least one of x, y, z is a multiple of 3.
   (b) Do part (a) again but with "3" replaced by "4", and then do it once more with "3" replaced by "5".

7. In class, it was claimed that for every pair of integers a, b, there are $x, y \in \mathbb{Z}$ with $ax + by = \gcd(a, b)$.

   The Euclidean algorithm gives a constructive proof of this theorem. We illustrate with the example of $x = 942$ and $y = 408$. Here the Euclidean algorithm runs as follows:

$$\begin{aligned} 942 &= 408 \cdot 2 + 126 \\ 408 &= 126 \cdot 3 + 30 \\ 126 &= 30 \cdot 4 + 6 \\ 30 &= 6 \cdot 5 + 0. \end{aligned}$$

   In particular, $\gcd(942, 408) = 6$. So there should be $x, y \in \mathbb{Z}$ with $942x + 408y = 6$.

   We can find x, y by backtracking through the algorithm. First,

$$6 = 126 + 30(-4), \qquad \text{so we get 6 as a combination of } 126, 30.$$

   Next,

$$\begin{aligned} 6 &= 126 + (408 - 126 \cdot 3)(-4) \\ &= 408(-4) + 126(13), \qquad \text{so we get 6 as a combination of } 408, 126. \end{aligned}$$

   Continuing,

$$\begin{aligned} 6 &= 408(-4) + (942 - 408 \cdot 2)(13) \\ &= 942 \cdot 13 + 408(-30), \qquad \text{so we get 6 as a combination of } 942, 408. \end{aligned}$$

   (a) Using this method, find integers x and y with $17x + 97y = \gcd(17, 97)$.

   (b) Find integers x and y with $161x + 63y = \gcd(161, 63)$.

8. Let n be a positive integer. Suppose that the decimal digits of n — read from right-to-left — are $a_0, a_1, \ldots, a_k$. Show that

$$n \equiv a_0 + a_1 + a_2 + a_3 + \cdots + a_k \pmod{9}.$$

   Use this to determine the remainder when 2022 is divided by 9.

9. (Fermat's little theorem again) Complete the proof from class that when p is prime, $a^p \equiv a \pmod{p}$ for **all** integers a. Remember that in class, we [will have] only handled the case when $a \in \mathbb{Z}^+$.

   Hint: Don't reinvent the wheel. Find a way to deduce the general result from the case handled in class.

10. Solve the following congruences.

   (a) $3x \equiv 2 \pmod{5}$

   (b) $243x + 17 \equiv 101 \pmod{725}$

   (c) $20x \equiv 30 \pmod{4}$

   (d) $15x \equiv 25 \pmod{35}$

MATH 6000 exercises

11(*). (a) Prove that there are infinitely many prime numbers.

(b) Prove that there are infinitely many prime numbers p satisfying $p \equiv 3 \pmod 4$.

12(*). Prove that there are infinitely many prime numbers p satisfying $p \equiv 3$ or $5 \pmod 8$.