The sum of divisors of n, modulo n

Paul Pollack
(joint work with Aria Anavi, Carl Pomerance, and Vladimir Shevelev)

UBC/SFU/UGA

June 17, 2012

Backstory

Let $\sigma(n) := \sum_{d|n} d$ denote the sum of the divisors of n. Thus, for example,

$$\sigma(14) = 1 + 2 + 7 + 14 = 24.$$

Many of the oldest problems in number theory can be considered attempts to better understand the behavior of $\sigma(n)$.

Backstory

Let $\sigma(n) := \sum_{d|n} d$ denote the sum of the divisors of n. Thus, for example,

$$\sigma(14) = 1 + 2 + 7 + 14 = 24.$$

Many of the oldest problems in number theory can be considered attempts to better understand the behavior of $\sigma(n)$.

Definition

A natural number n is called **perfect** if $\sigma(n) = 2n$ and **multiply perfect** if $\sigma(n) = kn$ for some k. In other words, n is multiply perfect if $\sigma(n) \equiv 0 \pmod{n}$.

For example, $n = 28$ is perfect (since $\sigma(n) = 56$) and $n = 120$ is multiply perfect (since $\sigma(120) = 360$).

We don't know if there are infinitely many perfect numbers or whether there are infinitely many multiply perfect numbers.

We have had better luck with upper bounds.

Theorem

We have the following estimates for $V(x)$, the number of perfect numbers up to x:

$$
\begin{aligned}
\textit{Volkmann, 1955} \quad & V(x) = O(x^{5/6}) \\
\textit{Hornfeck, 1955} \quad & V(x) = O(x^{1/2}) \\
\textit{Kanold, 1956} \quad & V(x) = o(x^{1/2}) \\
\textit{Erdős, 1956} \quad & V(x) = O(x^{1/2-\delta}) \\
\textit{Kanold, 1957} \quad & V(x) = O(x^{1/4}\frac{\log x}{\log\log x}) \\
\textit{Hornfeck \& Wirsing, 1957} \quad & V(x) = O(x^{\epsilon})
\end{aligned}
$$

The Hornfeck–Wirsing estimate hold also for the number of multiply perfect $n \leq x$.

Every 1 can make a difference

Definition

A natural number n is called **quasiperfect** (or **slightly excessive**) if $\sigma(n) = 2n + 1$. A number is called **multiply quasiperfect** if $\sigma(n) \equiv 1 \pmod{n}$.

Every 1 can make a difference

Definition

A natural number n is called **quasiperfect** (or **slightly excessive**) if $\sigma(n) = 2n + 1$. A number is called **multiply quasiperfect** if $\sigma(n) \equiv 1 \pmod{n}$.

Question

Can we show that the number of multiply quasiperfect $n \leq x$ is eventually smaller than x^ϵ?

Answer: No.

Every 1 can make a difference

Definition

A natural number n is called **quasiperfect** (or **slightly excessive**) if $\sigma(n) = 2n + 1$. A number is called **multiply quasiperfect** if $\sigma(n) \equiv 1 \pmod{n}$.

Question

Can we show that the number of multiply quasiperfect $n \leq x$ is eventually smaller than x^{ϵ}?

Answer: No. Every prime p satisfies $\sigma(p) = p + 1 \equiv 1 \pmod{p}$.

Every 1 can make a difference

Definition

A natural number n is called **quasiperfect** (or **slightly excessive**) if $\sigma(n) = 2n + 1$. A number is called **multiply quasiperfect** if $\sigma(n) \equiv 1 \pmod{n}$.

Question

Can we show that the number of multiply quasiperfect $n \leq x$ is eventually smaller than x^ϵ?

Answer: No. Every prime p satisfies $\sigma(p) = p + 1 \equiv 1 \pmod{p}$.

New question

Can we show that the number of **composite** *multiply quasiperfect $n \leq x$ is eventually smaller than x^ϵ?*

Theorem

The number of composite multiply quasi-perfect numbers up to x is at most

$$x^{1/2} \exp\left((2+o(1))\sqrt{\frac{\log x}{\log\log x}}\right).$$

Theorem (Anavi, P., Pomerance)

Consider the congruence $\sigma(n) \equiv a \pmod{n}$. If there is a multiply perfect number m with $\sigma(m) = a$, then every number $n = mp$ with $p \nmid m$ satisfies this congruence **(trivial solutions)**.

The number of solutions n to the congruence **not** *of this form* **(sporadic solutions)** *is at most*

$$x^{1/2+o(1)}, \qquad \text{as } x \to \infty,$$

uniformly for $|a| \le x^{1/4}$.

Messing with perfection

Definition

A natural number n is called **near-perfect** if n is the sum of all of its proper divisors except one of them, called the **redundant divisor**. Equivalently, n is **near-perfect** with **redundant divisor** d when

$$\sigma(n) = 2n + d, \quad \text{where } d \text{ is a proper divisor of } n.$$

Messing with perfection

Definition

A natural number n is called **near-perfect** if n is the sum of all of its proper divisors except one of them, called the **redundant divisor**. Equivalently, n is **near-perfect** with **redundant divisor** d when

$$\sigma(n) = 2n + d, \quad \text{where } d \text{ is a proper divisor of } n.$$

Example

196 is near-perfect with redundant divisor 7, since $\sigma(196) = 2 \cdot 196 + 7$.

The near-perfect numbers are (OEIS #A181595)
12, 18, 20, 24, 40, 56, 88, 104, 196, 224, 234, 368, 464, 650, 992, 1504, 1888, 1952, 3724, 5624, 9112, 11096, 13736, 15376, ...

We cannot prove that there are infinitely many near-perfect numbers, though we have certain Euclid-style families. For instance, if $M_p := 2^p - 1$ is prime, then

$$2^{p-1} M_p^2$$

is near-perfect with redundant divisor M_p.

We cannot prove that there are infinitely many near-perfect numbers, though we have certain Euclid-style families. For instance, if $M_p := 2^p - 1$ is prime, then

$$2^{p-1} M_p^2$$

is near-perfect with redundant divisor M_p.

In the opposite direction, we can prove the following:

Theorem (Anavi, P., Pomerance, Shevelev)

The number of near-perfect numbers in $[1, x]$ *is at most* $x^{3/4+o(1)}$, *as* $x \to \infty$.

Sketch of the proof

If $\sigma(n) = 2n + d$, then $\sigma(n) \equiv d \pmod{n}$. Moreover, n is a **sporadic** solution to this congruence.

For each $d \leq x^{1/4}$, we can apply our theorem to get an upper bound of $\approx x^{1/2}$ for each such d, and so an upper bound of $\approx x^{1/2} \cdot x^{1/4} = x^{3/4}$ total.

Suppose $d > x^{1/4}$. Since $d \mid n$ and $d \mid \sigma(n)$, we have $\gcd(n, \sigma(n)) \geq d > x^{1/4}$. Now we use the following theorem with $\alpha = \frac{1}{4}$.

Theorem (P.)

Fix $0 < \alpha < 1$. *The number of* $n \leq x$ *with* $\gcd(n, \sigma(n)) > x^{\alpha}$ *is* $x^{1-\alpha+o(1)}$.

A man's reach should exceed his grasp

Say that n is k-**nearly-perfect** if n is the sum of all its proper divisors with at most k exceptions.

- If $k = 1$, the k-nearly-perfects consist of the perfect numbers and the near-perfect numbers. The number of these up to x is at most $x^{3/4+o(1)}$. So we save a power of x over the trivial upper bound.

A man's reach should exceed his grasp

Say that n is k-**nearly-perfect** if n is the sum of all its proper divisors with at most k exceptions.

- If $k = 1$, the k-nearly-perfects consist of the perfect numbers and the near-perfect numbers. The number of these up to x is at most $x^{3/4+o(1)}$. So we save a power of x over the trivial upper bound.
- If $k \geq 4$, we don't save a power of x; this is because

$$6p = p + 2p + 3p$$

is 4-near-perfect for each $p > 3$.

A man's reach should exceed his grasp

Say that n is k-**nearly-perfect** if n is the sum of all its proper divisors with at most k exceptions.

- If $k = 1$, the k-nearly-perfects consist of the perfect numbers and the near-perfect numbers. The number of these up to x is at most $x^{3/4+o(1)}$. So we save a power of x over the trivial upper bound.
- If $k \geq 4$, we don't save a power of x; this is because

  $$6p = p + 2p + 3p$$

  is 4-near-perfect for each $p > 3$.
- **Problem:** What about $k = 2$ and $k = 3$?

A man's reach should exceed his grasp

Say that n is k-**nearly-perfect** if n is the sum of all its proper divisors with at most k exceptions.

- If $k = 1$, the k-nearly-perfects consist of the perfect numbers and the near-perfect numbers. The number of these up to x is at most $x^{3/4+o(1)}$. So we save a power of x over the trivial upper bound.
- If $k \geq 4$, we don't save a power of x; this is because

  $$6p = p + 2p + 3p$$

  is 4-near-perfect for each $p > 3$.
- **Problem:** What about $k = 2$ and $k = 3$?

One can also study n with **exactly** k redundant divisors. We can prove that for all large k, the counting function of such numbers grows at least as fast as $x/\log x$.

Thank you!