A remark on divisor weighted sums

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Let $\{a_n\}$ be a sequence of nonnegative real numbers. Under very mild hypotheses, we obtain upper bounds of the expected order of magnitude for sums of the form $\sum_{n\le x} a_n \tau_r(n)$, where $\tau_r(n)$ is the r-fold divisor function. This sharpens previous estimates of Friedlander and Iwaniec. The proof uses combinatorial ideas of Erdős and Wolke.

1 Introduction

Let $\mathscr{A} = \{a_n\}$ be a sequence of nonnegative real numbers, and let $A(x) = \sum_{n\le x} a_n$. In analytic number theory, one frequently encounters the problem of estimating sums of the form

$$D_r(x) := \sum_{n\le x} a_n \tau_r(n),$$

where $\tau_r(n) = \sum_{d_1\cdots d_r = n} 1$ is the r-fold divisor function. Since $\tau_r(n) \ll n^{\epsilon}$ (with an implied constant depending on r and ϵ), one has trivially that $D_r(x) \ll A(x)x^{\epsilon}$. However, this is quite crude, and it is desirable to find bounds closer to the expected order $A(x)(\log x)^{r-1}$. The following result in this direction was proved by Friedlander and Iwaniec in [3]. We let

$$A_d(x) := \sum_{\substack{n\le x\\ n\equiv 0 \pmod d}} a_n.$$

Proposition 1.1. *Let $x \ge 3$, and let k, r be integers with $k, r \ge 2$. Suppose that for all $d \le x^{1/k}$, we have $A_d(x) \ll A(x)g(d)$, where g is a nonnegative multiplicative function satisfying*

$$g(p^{\ell}) \ll p^{-\ell} \quad \text{for all primes } p \text{ and all } \ell \ge 1, \quad \text{and} \quad \prod_{p\le x}(1+g(p)) \ll \log x. \quad (1)$$

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Then

$$D_r(x) \ll A(x)(\log x)^{k^{(r-1)k}}.$$

The implied constant may depend on r, k, and the implicit constants in the assumptions on $A_d(x)$ and g.

Remark. Assuming that $\{a_n\}$ is supported on squarefree n, the same authors proved the sharper estimate $D_r(x) \ll A(x)(\log x)^{r^k/k}$.

In the arguments of [3], the fundamental components — as well as the main objects of study — are inequalities of the shape

$$\tau_r(n) \le c_{r,k} \sum_{\substack{d|n \\ d \le n^{1/k}}} f(d), \tag{2}$$

where $c_{r,k}$ is a constant depending only on r and k and f is a "small" multiplicative function. From this, one obtains an estimate of the form

$$\sum_{n \le x} a_n \tau_r(n) \ll A(x) \sum_{d \le x^{1/k}} f(d)g(d),$$

and the remaining sum on d can shown to be $(\log x)^{O(1)}$. For further discussion of inequalities of the type (2) (and some close relatives), see [12], [6], [2], [4], and [7].

In this note, we show how to prove an upper bound of the expected correct order, with no additional hypotheses. The components of the proof can all be traced back to work of Erdős [1] and Wolke [11]. In fact, by developing Erdős's methods, Wolke obtained results on partial sums of divisor-like functions over sequences satisfying very general sieve hypotheses. However, Wolke's set-up is sufficiently different that one cannot directly deduce an improvement of Proposition 1.1 from his results. So it seems that it may still be of some interest to have in the literature a nearly self-contained proof of the following estimate.

Theorem 1.2. *Under the same hypotheses as Proposition 1.1,*

$$D_r(x) \ll A(x)(\log x)^{r-1}.$$

The implied constant may depend on r, k, and the implicit constants in the assumptions on $A_d(x)$ and g.

A variant of this argument was recently used by the author to study the partial sums of $\tau_2(\#E(\mathbb{F}_p))$, where $E/\mathbb{Q}$ is a fixed non-CM elliptic curve and p runs over the primes of good reduction [8].

2 Proof of Theorem 1.2

We will assume throughout that x is sufficiently large in terms of r, k, and all of the implied constants in our assumptions — for if n is bounded in terms of these quantities, then the theorem follows from the trivial estimate $D_r(x) \le A(x) \cdot \max_{n \le x} \tau_r(n) \ll A(x)$.

For each $n \le x$, let us write $n = (p_1 \cdots p_j)(p_{j+1} \cdots p_J)$, where $p_1 \le p_2 \le \cdots \le p_J$ and j is chosen as large as possible with $p_1 \cdots p_j \le x^{1/k}$. Suppose to begin with that $J - j \le 2k$. It is elementary to check that τ_r is a submultiplicative function, i.e., $\tau_r(ab) \le \tau_r(a)\tau_r(b)$ for all positive integers a and b (one reference for this is [9]). Hence,

$$\begin{aligned}\tau_r(n) &\le (\tau_r(p_{j+1}) \cdots \tau_r(p_J))\tau_r(p_1 \cdots p_j)\\ &\le r^{2k}\tau_r(p_1 \cdots p_j) = r^{2k} \sum_{d_1 \cdots d_{r-1} \mid p_1 \cdots p_j} 1 \ll \sum_{\substack{d_1 \cdots d_{r-1} \mid n\\ d_1 \cdots d_{r-1} \le x^{1/k}}} 1.\end{aligned}$$

Thus, the sum of $a_n \tau_r(n)$ over these n is

$$\begin{aligned}&\ll \sum_{d_1 \cdots d_{r-1} \le x^{1/k}} A_{d_1 \cdots d_{r-1}}(x) \ll A(x) \sum_{d_1 \cdots d_{r-1} \le x^{1/k}} g(d_1 \cdots d_{r-1})\\ &\qquad \le A(x) \sum_{n \le x} g(n)\tau_{r-1}(n) \le A(x) \prod_{p \le x} \left(1 + (r-1)g(p) + \sum_{\ell \ge 2} g(p^\ell)\tau_{r-1}(p^\ell)\right).\end{aligned}$$

Since $g(p^\ell) \ll p^{-\ell}$ and $\tau_{r-1}(p^\ell) \ll p^{\ell/10}$ (say), we see that $\sum_p \sum_{\ell \ge 2} g(p^\ell)\tau_{r-1}(p^\ell) \ll 1$. Thus,

$$\begin{aligned}&\prod_{p \le x} \left(1 + (r-1)g(p) + \sum_{\ell \ge 2} g(p^\ell)\tau_{r-1}(p^\ell)\right)\\ &\qquad \le \left(\prod_{p \le x}(1 + g(p))\right)^{r-1} \exp\left(\sum_p \sum_{\ell \ge 2} g(p^\ell)\tau_{r-1}(p^\ell)\right) \ll (\log x)^{r-1}.\end{aligned}$$

Putting it all together, this part of the sum contributes $\ll A(x)(\log x)^{r-1}$, as desired.

Suppose now the integer $n \le x$ is such that $J - j > 2k$. Then $p_{j+1} < x^{\frac{1}{2k}}$, since otherwise $n \ge p_{j+1} \cdots p_J \ge p_{j+1}^{2k+1} > x$. Since $p_j < x^{\frac{1}{2k}}$, we can choose an integer $T \ge 2k$ with

$$x^{\frac{1}{T+1}} \le p_j < x^{\frac{1}{T}}.$$

Then $J - j \le T + 1$, and $\tau_r(p_{j+1} \cdots p_J) \le r^{T+1}$. From the pointwise bound $\tau_r(\cdot) \le \tau_2(\cdot)^{r-1}$ and the maximal order of the usual divisor function τ_2, we see that $\tau_r(p_{j+1} \cdots p_J) \le \exp(C \log x/\log\log x)$ for a certain constant C, depending only on r. Hence,

$$\tau_r(p_{j+1} \cdots p_J) \le \min\{r^{T+1}, \exp(C \log x/\log\log x)\} =: M_T.$$

By the choice of j,

$$p_1 \cdots p_j > x^{\frac{1}{k}}/p_{j+1} > x^{\frac{1}{2k}}.$$

By symmetry,

$$\tau_r(n) \le M_T \cdot \tau_r(p_1 \cdots p_j) \le r M_T \cdot \sum_{\substack{d_1 \cdots d_{r-1} \mid p_1 \cdots p_j\\ d_1 > x^{1/2kr}}} 1 \ll M_T \sum_{\substack{d_1 \cdots d_{r-1} \mid n\\ x^{1/2kr} < d_1 \cdots d_{r-1} \le x^{1/k}\\ p \mid d_1 \cdots d_{r-1} \Rightarrow p \le x^{1/T}}} 1.$$

We sum on n and then on T, keeping in mind that $A_{d_1\cdots d_{r-1}}(x) \ll A(x)g(d_1\cdots d_{r-1})$ whenever $d_1\cdots d_{r-1} \le x^{1/k}$. We find that the sum over the n not already accounted for above is

$$\ll A(x) \sum_{2k\le T\le \frac{\log x}{\log 2}} M_T \sum_{\substack{x^{1/2kr}<d_1\cdots d_{r-1}\le x^{1/k}\\ p\mid d_1\cdots d_{r-1}\Rightarrow p\le x^{1/T}}} g(d_1\cdots d_{r-1})$$

$$\le A(x) \sum_{2k\le T\le \frac{\log x}{\log 2}} M_T \sum_{\substack{x^{1/2kr}<m\le x^{1/k}\\ p\mid m\Rightarrow p\le x^{1/T}}} g(m)\tau_{r-1}(m). \quad (3)$$

We bound (3) by considering two ranges for T. Suppose first that T is very large, in the sense that $x^{1/T} \le (\log x)^2$. Since $g(p^\ell) \ll p^{-\ell}$, we see that $g(m) \le \tau_2(m)^{O(1)}/m$, and so $g(m) \le \exp(O(\log x/\log\log x))m^{-1}$ for all $m \le x$. An upper bound of the same shape holds for $\tau_{r-1}(m)$ and for M_T. Hence, this range of T contributes

$$\ll A(x)\cdot \log x \cdot \exp(O(\log x/\log\log x)) \cdot \sum_{\substack{m>x^{1/2kr}\\ p\mid m\Rightarrow p\le(\log x)^2}} \frac{1}{m}.$$

We can assume $\log x \ge (2kr)^5$. So if $t \ge x^{1/2kr}$, then $(\log t)^{5/2} \ge (\log x)^2$. Hence, every $m \le t$ with prime factors bounded by $(\log x)^2$ also has its prime factors bounded by $(\log t)^{5/2}$. From the theory of smooth numbers, the count of $m \le t$ with all prime factors bounded by $(\log t)^{5/2}$ is $t^{3/5+o(1)}$, as $t \to \infty$, and so is $O(t^{2/3})$ for all $t \ge 1$. (For a more general result, see [5, eq. (1.14)].) Thus,

$$\sum_{\substack{m>x^{\frac{1}{2kr}}\\ p\mid m\Rightarrow p\le(\log x)^2}} \frac{1}{m} = \sum_{\ell=0}^{\infty} \sum_{\substack{2^\ell x^{\frac{1}{2kr}}<m\le 2^{\ell+1}x^{\frac{1}{2kr}}\\ p\mid m\Rightarrow p\le(\log x)^2}} \frac{1}{m} \ll x^{-\frac{1}{6kr}} \sum_{\ell=0}^{\infty} 2^{-\ell/3} \ll x^{-\frac{1}{6kr}}.$$

Plugging this back in above, we see that this range of T contributes $\ll A(x)$, which is acceptable for us.

Now suppose that $x^{1/T} > (\log x)^2$. We argue as in the proof of [11, Lemma 3]. It will be convenient notationally to set

$$y = x^{1/T} \quad \text{and} \quad z = x^{1/2kr}.$$

For any positive $\eta < \frac{1}{3}$, we have (Rankin's trick)

$$\sum_{\substack{m>z\\ p\mid m\Rightarrow p\le y}} g(m)\tau_{r-1}(m) \le \sum_{m:\ p\mid m\Rightarrow p\le y} g(m)\tau_{r-1}(m)\left(\frac{m}{z}\right)^\eta$$

$$\le \exp\Bigg(-\eta\log z + (r-1)\sum_{p\le y} g(p)p^\eta + \sum_p \sum_{\ell\ge 2} g(p^\ell)p^{\ell\eta}\tau_{r-1}(p^\ell)\Bigg)$$

$$\ll \exp\Bigg(-\eta\log z + (r-1)\sum_{p\le y} g(p)p^\eta\Bigg);$$

the assumption that $\eta < \frac{1}{3}$ is used in moving from the second line to the third, to guarantee that the double sum on p and ℓ is $O(1)$. Now

$$\sum_{p \le y} g(p) p^{\eta} \le \sum_{p \le x} g(p) + \sum_{p \le y} g(p)(p^{\eta} - 1).$$

Using the inequality $\exp(t) = (1+t)\exp(O(t^2))$, valid for nonnegative and bounded t, we find that

$$\begin{aligned} \exp\left((r-1)\sum_{p \le x} g(p)\right) &= \exp\left(\sum_{p \le x} g(p)\right)^{r-1} \\ &\le \left(\prod_{p \le x}(1+g(p))\right)^{r-1} \exp\left(O\Big(\sum_{p} g(p)^2\Big)\right) \ll (\log x)^{r-1}. \end{aligned}$$

Also,

$$\sum_{p \le y} g(p)(p^{\eta} - 1) = \sum_{p \le y} g(p) \sum_{\ell=1}^{\infty} \frac{(\eta \log p)^{\ell}}{\ell!} \ll \sum_{\ell=1}^{\infty} \frac{\eta^{\ell}}{\ell!} \sum_{p \le y} \frac{(\log p)^{\ell}}{p}.$$

To deal with the inner sum, notice that for all $t \ge 2$, we have $\sum_{p \le t} (\log p)^{\ell} \le \pi(t)(\log t)^{\ell} \ll t(\log t)^{\ell-1}$. By partial summation, $\sum_{p \le y} \frac{(\log p)^{\ell}}{p} \ll (\log y)^{\ell}$. (Importantly, the implied constant here can be chosen independently of ℓ.) Inserting this above, we find that

$$\sum_{p \le y} g(p)(p^{\eta} - 1) \ll \sum_{\ell=1}^{\infty} \frac{(\eta \log y)^{\ell}}{\ell!} \le y^{\eta}.$$

Assembling what we have found so far,

$$\sum_{\substack{m > z \\ p \mid m \Rightarrow p \le y}} g(m)\tau_{r-1}(m) \ll \exp(O(y^{\eta}) - \eta \log z) \cdot (\log x)^{r-1}.$$

Choose $\eta = \frac{1}{3kr} \frac{T \log T}{\log z}$. Since $x^{1/T} > (\log x)^2$, we have $T < \frac{\log x}{2 \log \log x}$, and so

$$\eta = \frac{T \log T}{\frac{3}{2} \log x} < \frac{T \log \log x}{\frac{3}{2} \log x} < \frac{1}{3}.$$

Moreover, with choice of η, we have $\eta \log z = \frac{1}{3kr} T \log T$, while

$$y^{\eta} = \exp\left(\frac{\log y}{\frac{3}{2}\log x} T \log T\right) = \exp\left(\frac{2}{3} \log T\right) = T^{2/3}.$$

Hence,

$$\sum_{\substack{m > x^{1/2kr} \\ p \mid m \Rightarrow p \le x^{1/T}}} g(m)\tau_{r-1}(m) \ll \exp(O(T^{2/3})) \exp\left(-\frac{1}{3kr} T \log T\right) \cdot (\log x)^{r-1}.$$

Inserting this back into (3), and recalling that $M_T \le r^{T+1} = \exp(O(T))$, we find that the contribution from the remaining n is

$$\ll A(x)(\log x)^{r-1} \sum_{T} \exp(O(T)) \exp(O(T^{2/3})) \exp\left(-\frac{1}{3kr} T \log T\right).$$

The sum on T is $O(1)$, leading to an upper bound of $\ll A(x)(\log x)^{r-1}$. □

Remarks. The conditions on g can be somewhat relaxed: Suppose we replace the two conditions in (1) with the assumption that $g(n) \le \tau(n)^{O(1)}/n$. Minor modifications of our arguments will then show that

$$\sum_{n \le x} a_n \tau_r(n) \ll A(x) \exp((r-1) \sum_{p \le x} g(p)).$$

Moreover, it is clear from the proof of Theorem 1.2 that we did not need to take $A(x) = \sum_{n \le x} a_n$; all of the arguments hold if $A(x)$ is any upper bound on that sum. These two remarks should be useful in situations where the available upper bounds on $A_d(x)$ are somewhat crude.

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