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THE ERROR TERM IN THE COUNT OF ABUNDANT NUMBERS

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Abstract. A natural number n is called *abundant* if the sum of the proper divisors of n exceeds n. For example, 12 is abundant, since $1+2+3+4+6=16$. In 1929, Bessel-Hagen asked whether or not the set of abundant numbers possesses an asymptotic density. In other words, if $A(x)$ denotes the count of abundant numbers belonging to the interval $[1, x]$, does $A(x)/x$ tend to a limit? Four years later, Davenport answered Bessel-Hagen's question in the affirmative. Calling this density Δ, it is now known that $0.24761 < \Delta < 0.24766$, so that just under one in four numbers are abundant. We show that $A(x) - \Delta x < x/\exp((\log x)^{1/3})$ for all large x. We also study the behavior of the corresponding error term for the count of so-called α-*abundant numbers*.

§1. *Introduction.* Let $\sigma(n) := \sum_{d|n} d$ be the usual sum-of-divisors function. It is traditional to call the natural number n *abundant* if the sum of its proper divisors exceeds n, that is, if $\sigma(n) > 2n$. The abundant numbers have been of interest for over two thousand years. However, it was only comparatively recently, in 1929, that Bessel-Hagen [3, p. 1571] posed the question of whether or not the abundant numbers possess a natural density.

It did not take long for Bessel-Hagen's question to be answered. For $\alpha \geqslant 1$, we call an n satisfying $\sigma(n)/n \geqslant \alpha$ an α-*nondeficient number*, and we call an n satisfying $\sigma(n)/n > \alpha$ an α-*abundant number*. Inspired by earlier work of Schoenberg [27], Davenport [6] showed in 1933 that for every $\alpha \geqslant 1$, the set of α-nondeficient numbers possesses an asymptotic density $D(\alpha)$. Furthermore, $D(\alpha)$ is a continuous function of α and $D(\alpha) \to 0$ as $\alpha \to \infty$. These results were independently, and nearly simultaneously, obtained by Behrend (claimed in [2]) and Chowla [5]. For closely related results, see the papers of Erdős [9, 11, 12, 13] and Schoenberg [28]. This last paper contains a proof that $D(\alpha)$ is strictly decreasing for $\alpha \geqslant 1$.

Since $D(\alpha)$ is continuous, the solutions n to $\sigma(n)/n = \alpha$ comprise a set of asymptotic density zero for every fixed α. Thus, the α-abundant numbers have the same density $D(\alpha)$ as the α-nondeficient numbers.

Several authors have worked on the problem of obtaining numerical approximations of the density $D(2)$ of the abundant numbers. Notable efforts in this direction include those of Behrend [1], Salié [26], Wall [30], and Deléglise [7]. The current record is held by the first author [22], who proved

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in his thesis that $0.24761 < D(2) < 0.24766$. From this (or the earlier work of Deléglise), one sees that just under one in four numbers are abundant.

Here, we ask: what about the error? In other words, how close is $D(\alpha)x$ to the actual count of α-abundant numbers up to x?

For technical reasons, rather than discuss α-abundants directly, we will state our results in terms of α-nondeficient numbers. This is no cause for concern. Indeed, Wirsing [33] has shown that for all $x \geqslant 3$ and all $\alpha \geqslant 1$, the number of $n \leqslant x$ with $\sigma(n)/n = \alpha$ is at most $x^{W/\log\log x}$, where W is an absolute constant. This bound of $x^{W/\log\log x}$ will be minuscule in comparison with the error terms that appear later.

Let

$$A(\alpha;\, x) := \#\{\alpha\text{-nondeficient } n \leqslant x\},$$

and let

$$E(\alpha;\, x) = A(\alpha;\, x) - D(\alpha)x.$$

For the following estimate, which is a slight sharpening of a theorem of Fainleib [18], see Elliott [8, Theorem 5.6, p. 203]. We write $\log_k x$ for the kth iterate of the function $\log_1 x := \max\{1, \log x\}$.

THEOREM A. *For all $\alpha \geqslant 1$ and $x \geqslant 2$, we have*

$$E(\alpha;\, x) \ll \frac{x}{\log x} \cdot \frac{\log_2 x}{\log_3 x}.$$

Here the implied constant is absolute.

If one insists on uniformity in α, then Theorem A is almost the best possible. Indeed, it was known to Fainleib (compare with the footnote on [18, p. 860]) that $\sup_{\alpha \geqslant 1} |E(\alpha;\, x)| \gg x/\log x$ for all large x. A proof can be effected by considering the specific value $\alpha = 1 + 1/x$. For this α, every $n \in (1, x]$ is α-nondeficient, so that $A(\alpha;\, x) = x + O(1)$. On the other hand, a theorem of Erdős [14, Theorem 3] gives that $D(\alpha) = 1 - (1 + o(1))\mathrm{e}^{-\gamma}/\log x$ as $x \to \infty$. Thus, $|A(\alpha;\, x) - D(\alpha)x| = (1 + o(1))\mathrm{e}^{-\gamma} x/\log x$ as $x \to \infty$.

In the preceding example, α depends on x. One might hope that if one fixes α in advance, then a better error term is attainable. Our first theorem asserts that this is indeed the case for almost all α. Recall that α is called a *Liouville number* if α is irrational and, for every n, the inequality $|a/b - \alpha| < b^{-n}$ has a solution in integers a and b with $b > 1$. It is well known that the set of Liouville numbers has Lebesgue measure zero (see, for instance, [21, Theorem 198, p. 216]).

THEOREM 1.1. *Fix a non-Liouville number $\alpha \geqslant 1$. There are positive constants $x_0 = x_0(\alpha)$ and $C = C(\alpha)$ so that the following holds: for all $x > x_0$, we have*

$$E(\alpha;\, x) \ll x \exp(-C(\log x)^{1/3}(\log_2 x)^{2/3}).$$

We do not have nearly so strong a result in the Liouville case. However, we can establish a modest improvement to the Elliott–Fainleib theorem when α is fixed.

THEOREM 1.2. *Fix $\alpha \geqslant 1$. Then as $x \to \infty$,*

$$E(\alpha; x) = o\left(\frac{x}{\log x} \cdot \frac{\log_2 x}{\log_3 x}\right).$$

Our method is easily adapted to give a new proof of the Elliott–Fainleib theorem itself. While the arguments of Elliott and Fainleib depend heavily on the machinery of characteristic functions, our approach is entirely elementary and arithmetic, building on ideas introduced by Erdős to study primitive nondeficient numbers (defined in §2 below).

In our final theorem, we consider the problem of obtaining upper bounds for $|E(\alpha; x)|$ on average over α.

THEOREM 1.3. *As $x \to \infty$, we have*

$$\int_1^\infty |E(\alpha; x)|^2 \, d\alpha \leqslant x^2 \exp(-\sqrt{(1/2 + o(1)) \log x \log_2 x}).$$

Thus, for a given x, the set of $\alpha \geqslant 1$ where

$$|E(\alpha; x)| \geqslant x \exp\left(-\frac{1}{3\sqrt{2}}\sqrt{\log x \log_2 x}\right)$$

has measure at most

$$\exp\left(-\left(\frac{1}{3\sqrt{2}} + o(1)\right)\sqrt{\log x \log_2 x}\right) \quad \text{as } x \to \infty.$$

Similar results, slightly weaker and for $\varphi(n)/n$ in place of $\sigma(n)/n$, were proved by Fainleib (compare with [**17**, Theorem 2] and its proof). Our work largely follows his but with an additional optimization that allows us to introduce the double-logarithmic factor underneath the square root.

§2. *Preliminaries.*

2.1. *Primitive nondeficient numbers.* Call n *primitive α-nondeficient* if $\sigma(n)/n \geqslant \alpha$ but $\sigma(d)/d < \alpha$ for every d dividing n with $d < n$. Every α-nondeficient n possesses a primitive α-nondeficient divisor d; for instance, one can take d as the smallest α-nondeficient divisor of n. On the other hand, it is simple to show that every multiple of an α-nondeficient number is α-nondeficient. Thus, the α-nondeficient numbers are exactly those numbers possessing at least one primitive α-nondeficient divisor.

We will need the following result on the distribution of primitive α-nondeficient numbers, published by Erdős in 1958 (see [**15**, equations (4) and (5)]). The case $\alpha = 2$ is much older, and was found by Erdős already in 1935 [**10**].

LEMMA 2.1. *Fix a real number $\alpha \geqslant 1$.*

(i) *The count of primitive α-nondeficient numbers in $[1, x]$ is $o(x/\log x)$, as $x \to \infty$.*

(ii) *Suppose additionally that* α *is non-Liouville. Then there are positive constants* $K = K(\alpha)$ *and* $x_0 = x_0(\alpha)$ *so that for all* $x > x_0$*, the number of primitive* α*-nondeficient numbers not exceeding* x *is at most*

$$x/\exp(K\sqrt{\log x \log_2 x}).$$

Part (ii) of Lemma 2.1 is only asserted in [**15**], not proved. However, a detailed proof of this part of the lemma can be found in [**22**, Ch. 5].

2.2. *The fundamental decomposition.* In this section, we describe a convenient partition of the α-nondeficient numbers, due to the first author [**23**]. We start by introducing a total order $\preceq$ on the set of prime powers.

Definition 2.2. If p and q are primes and e and f are natural numbers, we say that $p^e \preceq q^f$ if either:

(i) $\sigma(p^e) < \sigma(q^f)$; or
(ii) $\sigma(p^e) = \sigma(q^f)$ and $p^e \leqslant q^f$.

In [**23**], the ordering $\preceq$ is called *the ordering with respect to decreasing significance*, since (in a sense made precise there) the prime power divisors of n that are small with respect to this ordering play the largest role in determining the size of $\sigma(n)/n$. The sequence of prime powers, listed in decreasing significance, begins

$$2,\ 3,\ 5,\ 2^2,\ 7,\ 11,\ 3^2,\ 13,\ 2^3,\ 17,\ 19,\ 23,\ 29,$$
$$2^4,\ 5^2,\ 31,\ 37,\ 3^3,\ 41,\ \ldots.$$

For each integer $s > 1$, we will write $P^*(s)$ for the prime power dividing s which is largest with respect to the ordering $\preceq$.

Now suppose that $\alpha > 1$ is fixed. Let $\mathscr{S}$ be the set of primitive α-nondeficient numbers, and let $\mathscr{A}$ be the set of all α-nondeficient numbers. For each $s \in \mathscr{S}$, let

$$L_s := \operatorname*{lcm}_{p^e \preceq P^*(s)} [p^e]. \tag{2.1}$$

The maximality of $P^*(s)$ guarantees that $s | L_s$. The following proposition is a restatement of the main theoretical result of [**23**] (compare with that paper's Theorem 2 and Corollary 1). We use the symbol $\dot\cup$ to assert that the sets appearing in a union are disjoint.

PROPOSITION 2.3. *For each* $s \in \mathscr{S}$*, let* $\mathscr{A}_s = \{sq : q \in \mathbf{N}$ *and* $\gcd(q, L_s/s) = 1\}$*. Then*

$$\mathscr{A} = \dot{\bigcup_{s\in\mathscr{S}}} \mathscr{A}_s. \tag{2.2}$$

Let $D(\mathscr{A}_s)$ *denote the asymptotic density of* $\mathscr{A}_s$*. Then for the density* $D(\alpha)$ *of* $\mathscr{A}$*, we have*

$$D(\alpha) = \sum_{s\in\mathscr{S}} D(\mathscr{A}_s) = \sum_{s\in\mathscr{S}} \frac{1}{s} \prod_{p|(L_s/s)} (1 - 1/p). \tag{2.3}$$

2.3. *Analytic tools.* Our arguments utilize certain results from the standard tool chest of analytic and probabilistic number theory. The first of these (which appears as [20, Exercise 05, p. 12]) bounds from above the number of $n \leqslant x$ with abnormally many prime factors. We write $\Omega(n) = \sum_{p^k \| n} k$ for the number of primes dividing n, counted with multiplicity.

LEMMA 2.4. *Let $x \geqslant 2$, and let $k \geqslant 1$. The count of natural numbers $n \leqslant x$ with $\Omega(n) \geqslant k$ is*

$$\ll \frac{k}{2^k} x \log x.$$

For a detailed proof of Lemma 2.4, see [24, Lemmas 12 and 13].

We also need a fairly sharp upper bound on the count of smooth numbers. For each natural number n, we let $P(n)$ denote the largest prime factor of n, with the convention that $P(1) = 1$. We say n is *y-smooth* if $P(n) \leqslant y$, and we let $\Psi(x, y)$ denote the count of y-smooths in the interval $[1, x]$. The following estimate is due to de Bruijn [4].

LEMMA 2.5. *Suppose that $x \geqslant y \geqslant 2$, and write $u := \log x / \log y$. Then as $u \to \infty$, we have*

$$\Psi(x, y) \leqslant x \exp(-(1 + o(1)) u \log u),$$

uniformly in the range $y \geqslant (\log x)^2$.

We will appeal to the following special case of the fundamental lemma of the sieve (see [19, Theorem 2.5, p. 82]).

LEMMA 2.6. *Let $x \geqslant 3$, and let $\mathscr{P}$ be a set of primes contained in $[2, z]$. Suppose that $z \leqslant x$. Then with $u := (\log x)/(\log z)$, we have*

$$\sum_{\substack{n \leqslant x \\ p \mid n \Rightarrow p \notin \mathscr{P}}} 1 = \left(x \prod_{p \in \mathscr{P}} (1 - 1/p) \right) \left(1 + O\left(\exp\left(-\frac{1}{2} u \log u \right) \right) \right.$$
$$\left. + O(\exp(-\sqrt{\log x})) \right).$$

Finally, it will be convenient to have at hand a result of Wintner describing the mean values of certain well-behaved arithmetic functions (see [29, Corollary 2.2, p. 50]).

LEMMA 2.7. *Let f be an arithmetic function, and choose h so that $f(n) = \sum_{d \mid n} h(d)$ for every natural number n. Suppose that $\sum_{d=1}^{\infty} |h(d)|/d < \infty$. Then as $N \to \infty$,*

$$\frac{1}{N} \sum_{n \leqslant N} f(n) \to \sum_{d=1}^{\infty} \frac{h(d)}{d}.$$

§3. *Proof of Theorem 1.1.* If $\alpha = 1$, then Theorem 1.1 is trivial, and so we suppose that $\alpha > 1$ is fixed and non-Liouville. We let $\mathscr{S}$ be the set of primitive α-nondeficient numbers, and we let $\mathscr{A}$ be the set of all α-nondeficient numbers. For the rest of this argument, we let

$$y := \exp((\log x)^{2/3}(\log_2 x)^{1/3}).$$

Intersecting the decomposition (2.2) with $[1, x]$, we can write $\mathscr{A} \cap [1, x] = \dot{\bigcup}_{s\in\mathscr{S}} \mathscr{A}_s$, where now each $\mathscr{A}_s := \{sq : q \leqslant x/s : \gcd(q, L_s/s) = 1\}$. Thus,

$$A(\alpha; x) = \left|\dot{\bigcup_{\substack{s\in\mathscr{S}\\ s>y}}} \mathscr{A}_s\right| + \sum_{\substack{s\in\mathscr{S}\\ s\leqslant y}} |\mathscr{A}_s|.$$

Using the upper bound of Lemma 2.1(ii) for the counting function of $\mathscr{S}$, along with partial summation, we find that

$$\begin{aligned}\left|\dot{\bigcup_{\substack{s\in\mathscr{S}\\ s>y}}} \mathscr{A}_s\right| \leqslant x \sum_{\substack{s\in\mathscr{S}\\ s>y}} \frac{1}{s} &\leqslant x \exp\left(-\frac{K}{2}\sqrt{\log y \log_2 y}\right)\\ &\leqslant x \exp\left(-\frac{K}{3}(\log x)^{1/3}(\log_2 x)^{2/3}\right), \qquad (3.1)\end{aligned}$$

which may be considered to belong to the error term of the theorem. Moreover, we may write

$$\sum_{\substack{s\in\mathscr{S}\\ s\leqslant y}} |\mathscr{A}_s| = \sum_{\substack{s\in\mathscr{S}\\ s\leqslant y}} \sum_{\substack{q\leqslant x/s\\ \gcd(q,L_s/s)=1}} 1. \qquad (3.2)$$

Each of the inner sums is estimated by the fundamental lemma of the sieve. Suppose that $s \in \mathscr{S}$ with $s \leqslant y$. As in §2, let $P^*(s)$ be the largest prime power dividing s with respect to $\preceq$. If p is a prime dividing L_s, then $p \preceq P^*(s)$; consequently,

$$p < \sigma(p) \leqslant \sigma(P^*(s)) \leqslant 2P^*(s) \leqslant 2s \leqslant 2y.$$

Thus, estimating the inner sum in (3.2) amounts to sieving $[1, x/s]$ by the set $\mathscr{P} := \{p | L_s/s\}$, which is a set of primes contained in $[2, 2y]$. With $u := \log(x/s)/\log(2y)$, Lemma 2.6 gives

$$\begin{aligned}\sum_{\substack{q\leqslant x/s\\ \gcd(q,L_s/s)=1}} 1 &= \frac{x}{s}\prod_{p|(L_s/s)}(1-1/p)\left(1 + O\left(\exp\left(-\frac{1}{2}u\log u\right)\right)\right.\\ &\qquad \left. + O(\exp(-\sqrt{\log (x/s)}))\right)\\ &= \frac{x}{s}\prod_{p|(L_s/s)}(1-1/p)\\ &\qquad \times \left(1 + O\left(\exp\left(-\frac{1}{10}(\log x)^{1/3}(\log_2 x)^{2/3}\right)\right)\right).\end{aligned}$$

Inserting this estimate into (3.2), we find that

$$\Bigg|\dot{\bigcup_{\substack{s\in\mathscr{S}\\ s\leqslant y}}}\mathscr{A}_s\Bigg| = x\sum_{\substack{s\in\mathscr{S}\\ s\leqslant y}}\frac{1}{s}\prod_{p|(L_s/s)}(1-1/p) + O\bigg(x\exp\bigg(-\frac{1}{10}(\log x)^{1/3}(\log_2 x)^{2/3}\bigg)\bigg).$$

Moreover, from (2.3),

$$x\sum_{\substack{s\in\mathscr{S}\\ s\leqslant y}}\frac{1}{s}\prod_{p|(L_s/s)}(1-1/p) = D(\alpha)x - x\sum_{\substack{s\in\mathscr{S}\\ s>y}}\frac{1}{s}\prod_{p|(L_s/s)}(1-1/p) = D(\alpha)x + O\bigg(x\exp\bigg(-\frac{K}{3}(\log x)^{1/3}(\log_2 x)^{2/3}\bigg)\bigg),$$

using in the last step the upper bound (3.1) on $x\sum_{s\in\mathscr{S},s>y}1/s$. Collecting everything, and recalling that $E(\alpha;x) = A(\alpha;x) - D(\alpha)x$, we obtain the theorem with $C = \min\{K/3, 1/10\}$.

§4. *Proof of Theorem 1.2*. We begin by assuming that $\alpha > 1$ is fixed. We use $\mathscr{S}$, $\mathscr{A}$, and $\mathscr{A}_s$ with the same meanings as in the proof of Theorem 1.1. Throughout this argument, we assume that y is defined by

$$y := \exp\bigg(\frac{1}{10}\log x\frac{\log_3 x}{\log_2 x}\bigg).$$

We will always assume, sometimes without comment, that x is sufficiently large.

Applying Lemma 2.6 as in the proof of Theorem 1.1, we see that with $u_s := \log(x/s)/\log(2P^*(s))$,

$$A(\alpha;x) - \sum_{y<s\leqslant x}|\mathscr{A}_s| = \sum_{\substack{s\in\mathscr{S}\\ s\leqslant y}}\sum_{\substack{q\leqslant x/s\\ \gcd(q,L_s/s)=1}}1 = \sum_{\substack{s\in\mathscr{S}\\ s\leqslant y}}\frac{x}{s}\prod_{p|(L_s/s)}(1-1/p)\bigg(1+O\bigg(\exp\bigg(-\frac{1}{2}u_s\log u_s\bigg)\bigg) + O(\exp(-\sqrt{\log(x/s)}))\bigg).$$

Let us estimate the O-terms appearing above. Setting $u := \log(x/y)/\log(2y)$, each $u_s \geqslant u$. A short computation gives $\exp(-\frac{1}{2}u\log u) \ll (\log x)^{-4}$. Moreover, for $s \leqslant y$, we have $\exp(-\sqrt{\log(x/s)}) \leqslant \exp(-\frac{1}{2}\sqrt{\log x}) \ll (\log x)^{-4}$. Using these estimates, we derive that

$$A(\alpha;x) = \sum_{\substack{s\in\mathscr{S}\\ s\leqslant y}}\frac{x}{s}\prod_{p|(L_s/s)}(1-1/p) + \sum_{\substack{s\in\mathscr{S}\\ y<s\leqslant x}}|\mathscr{A}_s| + O(x(\log x)^{-4}).$$

Thus, from (2.3),

$$\begin{aligned} E(\alpha; x) &= A(\alpha; x) - D(\alpha)x \\ &= A(\alpha; x) - x \sum_{s\in\mathscr{S}} \frac{1}{s} \prod_{p\mid(L_s/s)} (1-1/p) \\ &= -\sum_{\substack{s\in\mathscr{S}\\ s>y}} \frac{x}{s} \prod_{p\mid(L_s/s)} (1-1/p) + \sum_{\substack{s\in\mathscr{S}\\ y<s\leqslant x}} |\mathscr{A}_s| + O(x(\log x)^{-4}). \end{aligned} \tag{4.1}$$

When $u_s = \log(x/s)/\log(2P^*(s))$ is sufficiently large, Lemma 2.6 yields $|\mathscr{A}_s| \asymp (x/s)\prod_{p\mid(L_s/s)}(1-1/p)$. In particular, this estimate holds for all $s \leqslant x^\delta$, for a certain (small) absolute constant $\delta > 0$. Using this estimate for the values of $s \in (y, x^\delta]$ that appear in the first sum in (4.1), we deduce that

$$\begin{aligned} E(\alpha; x) = O\Bigg(x \sum_{\substack{s\in\mathscr{S}\\ s>x^\delta}} \frac{1}{s} \prod_{p\mid(L_s/s)} (1-1/p)\Bigg) \\ + O\Bigg(\sum_{\substack{s\in\mathscr{S}\\ y<s\leqslant x}} |\mathscr{A}_s|\Bigg) + O(x(\log x)^{-4}). \end{aligned} \tag{4.2}$$

The final O-term in (4.2) is negligible, and the remainder of the proof is devoted to showing that the first two O-terms on the right-hand side are both $o((x/\log x)(\log_2 x/\log_3 x))$, as $x \to \infty$.

To continue, we discard certain inconvenient values of s. Let $p(n)$ be the smallest prime factor of n, with the convention that $p(1) = \infty$. Recall that $P(n)$ denotes the largest prime factor of n.

Definition 4.1. Call $s \in \mathscr{S}$ *typical* if all of the following hold:

(i) $\Omega(s) < 10\log_2 s$;
(ii) $P(s) > \exp(\frac{1}{10}\log s \log_3 s/\log_2 s)$;
(iii) the largest squarefull divisor of s is smaller than $(\log s)^6$;
(iv) if we write $s = ab$, where $P(a) \leqslant (\log s)^3$ and $p(b) > (\log s)^3$, then

$$p(b) > \exp(\tfrac{1}{30}\log s \log_3 s/\log_2 s).$$

LEMMA 4.2. *The count of atypical $s \in \mathscr{S} \cap [1, w]$ is $O(w/(\log w)^3)$ for all $w \geqslant 2$. Here the implied constant is absolute.*

Proof. Summing dyadically, it is enough to show that for all w exceeding a certain absolute constant, the count of atypical $s \in (w/2, w]$ is $O(w/(\log w)^3)$. If $s \in (w/2, w]$ fails condition (i), then $\Omega(s) \geqslant k$ where $k := \lceil 10\log_2(w/2)\rceil$. By Lemma 2.4, the number of these $s \in (w/2, w]$ is $\ll (k/2^k)w\log w \ll w/(\log w)^5$. If (ii) fails for $s \in (w/2, w]$, then $P(s) \leqslant \exp(\frac{1}{10}\log w \log_3 w/\log_2 w)$, and by Lemma 2.5, this places s into a set of size $O(w/(\log w)^9)$. The number of $s \in (w/2, w]$ where (iii) fails is

$$\leqslant w \sum_{\substack{n \geqslant (\log(w/2))^6 \\ n \text{ squarefull}}} \frac{1}{n} \ll \frac{w}{(\log w)^3};$$

here we have applied partial summation, noting that the count of squarefull numbers in $[1, t]$ is $O(t^{1/2})$ for all $t \geqslant 1$.

The remainder of the proof consists in bounding from above the number of $s \in (w/2, w]$ satisfying (i)–(iii) but failing (iv).

For each such s, write $s = ab$ with $P(a) \leqslant (\log s)^3$ and $p(b) > (\log s)^3$. Note that (iii) implies that b is squarefree. We have

$$a \leqslant P(a)^{\Omega(a)} \leqslant \exp(30(\log_2 s)^2) \leqslant \exp(30(\log_2 w)^2),$$

and since $s = ab > w/2$, it must be that $b > 1$ (since we are assuming w is large). Let $p = p(b)$, so that $p > (\log s)^3 > \frac{1}{2}(\log w)^3$. We claim that the integers s/p are all distinct. Since each $s/p < 2w/(\log w)^3$, it will follow that there are only $O(w/(\log w)^3)$ values of s satisfying (i)–(iii) but failing (iv), and so the proof of the lemma will be complete.

To prove the claim, suppose for the sake of contradiction that $s_1/p_1 = s_2/p_2$ where $s_1 \neq s_2$. Then $\sigma(s_1/p_1)/(s_1/p_1) = \sigma(s_2/p_2)/(s_2/p_2)$. Since each $p_i \parallel s_i$, we have $\sigma(s_i/p_i) = \sigma(s_i)/(p_i + 1)$, and so upon rearranging, we obtain

$$\frac{\sigma(s_1)}{s_1}\left(\frac{\sigma(s_2)}{s_2}\right)^{-1} = \frac{(p_1+1)p_2}{(p_2+1)p_1}.$$

Clearly, $p_1 \neq p_2$, and so we can assume without loss of generality that $p_2 > p_1$. Then the right-hand fraction exceeds 1, and so in fact

$$\begin{aligned} \frac{\sigma(s_1)}{s_1}\left(\frac{\sigma(s_2)}{s_2}\right)^{-1} &\geqslant 1 + \frac{1}{(p_2+1)p_1} \geqslant 1 + \frac{1}{2p_1p_2} \\ &\geqslant 1 + \frac{1}{2\exp(\frac{1}{15}\log w \log_3 w/\log_2 w)}. \end{aligned} \tag{4.3}$$

We used here that each $p_i \leqslant \exp(\frac{1}{30}\log s_i \log_3 s_i/\log_2 s_i) \leqslant \exp(\frac{1}{30} \log w \log_3 w/\log_2 w)$, by our assumption that (iv) fails for s_i. Now let P denote the largest prime factor of s_1. Since s_1 is primitive α-nondeficient, we have $\sigma(s_1/P)/(s_1/P) < \alpha$, and so

$$\frac{\sigma(s_1)}{s_1} = \frac{\sigma(s_1/P)}{s_1/P}\frac{\sigma(P)}{P} < \alpha\left(1 + \frac{1}{P}\right).$$

Thus,

$$\frac{\sigma(s_1)}{s_1}\left(\frac{\sigma(s_2)}{s_2}\right)^{-1} < \alpha\left(1 + \frac{1}{P}\right)\cdot\alpha^{-1} = 1 + \frac{1}{P}. \tag{4.4}$$

Comparing (4.3) and (4.4), we see that

$$P < 2\exp(\tfrac{1}{15}\log w \log_3 w/\log_2 w).$$

But for large w, this contradicts (ii) (remember that $s > w/2$). This argument is essentially due to Erdős [**15**, pp. 28–29]. □

Recall the definition of L_s given in (2.1).

LEMMA 4.3. *Suppose that s is typical, in the sense of Definition 4.1. Then, provided that s exceeds a certain absolute constant, we have*

$$\prod_{p|(L_s/s)} (1-1/p) \ll \frac{1}{\log P(s)}. \tag{4.5}$$

Proof. If $p \leqslant P(s)$ is prime, then $p \preceq P(s) = P^*(s)$. Hence, $\prod_{p\leqslant P(s)} p | L_s$. Moreover, every prime $p \leqslant (\log s)^3$ divides L_s/s. Indeed, for $p \leqslant (\log s)^3$, we may choose an $e \geqslant 2$ with $(\log s)^6 \leqslant p^e < (\log s)^9$. By condition (iii) of typicality, $p^e \nmid s$. On the other hand, the lower bound (ii) on $P(s)$ for typical s shows that

$$\sigma(p^e) < 2p^e < 2(\log s)^9 < P(s) < \sigma(P(s)).$$

Thus, $p^e \preceq P(s) = P^*(s)$, and so $p^e | L_s$. Since p divides L_s to a higher power than that to which it divides s, we get that $p | L_s/s$.

Putting together the observations of the last paragraph, we see that every prime $p \leqslant P(s)$ divides L_s/s except possibly those prime divisors of s exceeding $(\log s)^3$. Since $\Omega(s) < 10\log_2 s$, we find that

$$\begin{aligned}
\prod_{p|(L_s/s)} \left(1-\frac{1}{p}\right) &\leqslant \prod_{p\leqslant P(s)} \left(1-\frac{1}{p}\right) \prod_{\substack{p|s\\ p>(\log s)^3}} \left(1-\frac{1}{p}\right)^{-1} \\
&\ll \frac{1}{\log P(s)} \prod_{\substack{p|s\\ p>(\log s)^3}} \left(1+\frac{1}{p}\right) \\
&\leqslant \frac{1}{\log P(s)} \left(1+\frac{1}{(\log s)^3}\right)^{10\log_2 s} \ll \frac{1}{\log P(s)},
\end{aligned}$$

as was to be proved. □

We now continue the proof of the main theorem. Referring back to (4.2), what remains to be shown is that both expressions

$$x \sum_{\substack{s\in\mathscr{S}\\ s>x^\delta}} \frac{1}{s} \prod_{p|(L_s/s)} (1-1/p) \tag{4.6}$$

and

$$\sum_{\substack{s\in\mathscr{S}\\ y<s\leqslant x}} |\mathscr{A}_s| \tag{4.7}$$

are $o((x/\log x)\log_2 x/\log_3 x)$, as $x \to \infty$. It is comparatively simple to dispense with (4.6). By Lemma 4.2 and partial summation, the contribution to (4.6) from atypical s is at most $x\sum_{\text{atypical } s>x^{\delta}} 1/s \ll x/(\log x)^2$. This is negligible. For the remaining values of s, we use (4.5) and condition (ii) of typicality to deduce that

$$x \sum_{\substack{s\in\mathscr{S} \\ s>x^{\delta} \\ s \text{ typical}}} \frac{1}{s} \prod_{p\mid(L_s/s)} (1-1/p) \ll x \sum_{\substack{s\in\mathscr{S} \\ s>x^{\delta} \\ s \text{ typical}}} \frac{1}{s\log P(s)} \ll x \sum_{\substack{s\in\mathscr{S} \\ s>x^{\delta}}} \frac{1}{s\log s}\frac{\log_2 s}{\log_3 s}.$$

By Lemma 2.1(i) and partial summation, this last expression is $o((x/\log x)\log_2 x/\log_3 x)$ as $x \to \infty$, as desired.

The treatment of (4.7) is more intricate. We again draw on the ideas of Erdős (compare with [**15**, pp. 31–32]). The contribution to (4.7) from atypical s is trivially bounded by $x\sum_{\text{atypical } s>y} 1/s \ll x/(\log y)^2 \ll x/(\log x)^{3/2}$. So we need only consider the contribution from typical $s \in (y, x]$. Recalling condition (iv) in Definition 4.1, we write each such s in the form $s = ab$ with $P(a) \leqslant (\log s)^3$ and $p(b) \geqslant \exp(\frac{1}{30}\log s \log_3 s/\log_2 s)$. We will use this meaning of a and b for the remainder of the argument.

LEMMA 4.4. *For all typical $s \in (y, x]$, the fraction $\sigma(a)/a$ assumes the same value.*

Proof. Suppose that it does not. Then there are $s_1, s_2 \in (y, x]$ with decompositions $s_i = a_i b_i$ so that $\sigma(a_1)/a_1 > \sigma(a_2)/a_2$. Since each $\Omega(s_i) \leqslant 10\log_2 s_i$, each

$$a_i \leqslant P(a_i)^{\Omega(a_i)} \leqslant \exp(30(\log_2 s_i)^2) \leqslant \exp(30(\log_2 x)^2).$$

Consequently,

$$\frac{\sigma(a_1)}{a_1} - \frac{\sigma(a_2)}{a_2} \geqslant \frac{1}{a_1 a_2} \geqslant \frac{1}{\exp(60(\log_2 x)^2)},$$

so that

$$\frac{\sigma(a_2)}{a_2} \leqslant \frac{\sigma(a_1)}{a_1} - \frac{1}{\exp(60(\log_2 x)^2)} \leqslant \alpha - \frac{1}{\exp(60(\log_2 x)^2)}.$$

Hence,

$$\begin{aligned}\frac{\sigma(s_2)}{s_2} = \frac{\sigma(a_2)}{a_2}\frac{\sigma(b_2)}{b_2} &\leqslant \frac{\sigma(a_2)}{a_2}\left(1+\frac{1}{p(b_2)}\right)^{\Omega(b_2)} \\ &\leqslant \left(\alpha - \frac{1}{\exp(60(\log_2 x)^2)}\right) \\ &\quad\times\left(1+\frac{1}{\exp(\frac{1}{30}\log y\frac{\log_3 y}{\log_2 y})}\right)^{10\log_2 x}.\end{aligned}$$

As shown by Gronwall, $\limsup_{n\to\infty} \sigma(n)/n\log_2 n = \mathrm{e}^{\gamma}$, where γ is the Euler–Mascheroni constant (see [21, Theorem 323, p. 350]). It follows that $\alpha \leqslant \sigma(s_1)/s_1 < 2\log_2 x$ (assuming x is sufficiently large). Now recalling the definition of y, we see that the final expression in the previous display is

$$\leqslant \alpha\left(1 - \frac{1}{\exp(60(\log_2 x)^2)\alpha}\right)\left(1 + \frac{1}{\exp\left(\frac{1}{301}\log x(\log_3 x)^2/(\log_2 x)^2\right)}\right)^{10\log_2 x}$$
$$\leqslant \alpha\left(1 - \frac{1}{\exp(61(\log_2 x)^2)}\right)\left(1 + \frac{1}{\exp\left(\frac{1}{400}\log x(\log_3 x)^2/(\log_2 x)^2\right)}\right),$$

which is smaller than α. But this contradicts that s is α-nondeficient. □

We now define

$$\mathscr{S}_1 := \{\text{typical } s \in \mathscr{S} \cap (y, x] \text{ with } b \text{ prime}\},$$
$$\mathscr{S}_2 := \{\text{typical } s \in \mathscr{S} \cap (y, x] \text{ with } b \text{ composite}\}.$$

We estimate separately the contribution to (4.7) from n belonging to $\mathscr{A}_s$ for $s \in \mathscr{S}_1$ versus $s \in \mathscr{S}_2$.

When $s \in \mathscr{S}_1$. Suppose that $n \in \mathscr{A}_s$ for a certain $s = ab \in \mathscr{S}_1$. Then $b = p$ is prime, and

$$p = s/a \geqslant s/\exp(30(\log_2 s)^2) > s^{1/2} > y^{1/2}.$$

Since $n \in \mathscr{A}_s$, if we write $n = sq = apq$, then q is coprime to L_s/s. The proof of Lemma 4.3 shows that L_s/s is divisible by every prime not exceeding $P(s)$, except possibly those dividing s and exceeding $(\log s)^3$. In our case, $p = P(s)$ is the prime divisor of s greater than $(\log s)^3$. Hence, if p' is the least prime dividing q, then $p' \geqslant p > y^{1/2}$.

Consequently, every $n \in \bigcup_{s\in\mathscr{S}_1} \mathscr{A}_s$ can be written in the form $n = aQ$ where $p(Q) > y^{1/2}$. Fixing a, applying the sieve once more shows that the number of $n \leqslant x$ of this form is $O(x/(a\log y))$. Thus, the total number of n belonging to $\mathscr{A}_s$ for some $s \in \mathscr{S}_1$ is

$$\ll \frac{x}{\log y}\sum_a \frac{1}{a}. \tag{4.8}$$

We claim that the sum on a is $o(1)$ as $x \to \infty$. This will show that the number of n in question is $o(x/\log y)$, which is $o((x/\log x)\log_2 x/\log_3 x)$ from the definition of y.

To derive the estimate for the sum, we appeal to the following lemma.

LEMMA 4.5. *Let β be an arbitrary real number. Then $\sum_{a>z,\sigma(a)/a=\beta} 1/a \to 0$ as $z \to \infty$, uniformly in the choice of β.*

Proof. This follows from Wirsing's theorem [33] that for $t \geqslant 3$, the number of $n \in [1, t]$ with $\sigma(n)/n = \beta$ is at most $t^{W/\log_2 t}$ for a certain absolute constant W. In fact, using Wirsing's estimate together with partial summation shows that the sum in the lemma statement is $O_\epsilon(z^{-1+\epsilon})$, as $z \to \infty$, uniformly in β. □

We know from Lemma 4.4 that $\sigma(a)/a$ assumes the same value β (say) for all a appearing in the sum (4.8). So from Lemma 4.5, if we show that the minimal value of a corresponding to an $s \in \mathscr{S}_1$ tends to infinity with x, then the sum in (4.8) is indeed $o(1)$. We proceed by contradiction. If the values of a do not tend to infinity, then there are infinitely many α-primitive nondeficient numbers of the form $s = a_0 p$, where a_0 is *fixed* and p is a prime not dividing a_0. For each of these values of s,

$$\frac{\sigma(a_0 p)}{a_0 p} = \frac{\sigma(a_0)}{a_0}(1 + 1/p) \geqslant \alpha.$$

Since p tends to infinity with s here, it must be that $\sigma(a_0)/a_0 \geqslant \alpha$. In other words, a_0 is α-nondeficient. But then $a_0 p$ is not *primitive* α-nondeficient for any prime p. This is a contradiction.

When $s \in \mathscr{S}_2$. Finally, suppose that $n \in \mathscr{A}_s$ for an $s \in \mathscr{S}_2$. We show that all such n number at most $O(x/(\log x)^{3/2})$, which is negligible. Let

$$p_0 = \min\{p(b) : s \in \mathscr{S}_2\}.$$

Since each $s \in \mathscr{S}_2$ is typical and larger than y,

$$p_0 \geqslant \exp\left(\frac{1}{30} \log y \frac{\log_3 y}{\log_2 y}\right) > \exp((\log x)^{0.9}), \tag{4.9}$$

say. We take two cases. Suppose first that $n \in \mathscr{A}_s$ for some $s \in \mathscr{S}_2$ having

$$p(b) \leqslant p_0(1 + 1/\log x).$$

Then n itself has a prime divisor in the interval $[p_0, p_0(1 + 1/\log x)]$. The number of these $n \leqslant x$ is at most

$$x \sum_{\substack{p \text{ prime} \\ p_0 \leqslant p \leqslant p_0(1+1/\log x)}} \frac{1}{p} \leqslant \frac{x}{p_0} \sum_{\substack{p \text{ prime} \\ p_0 \leqslant p \leqslant p_0(1+1/\log x)}} 1$$
$$\ll \frac{x}{p_0} \cdot \frac{p_0/\log x}{\log(p_0/\log x)} \ll \frac{x}{\log x \log p_0} \ll \frac{x}{(\log x)^{1.9}},$$

using the Brun–Titchmarsh inequality for primes in short intervals in the second step (see [**19**, Theorem 3.7, p. 107]) and the lower bound (4.9) in the last step. This estimate is acceptable for us.

In the second case, we suppose that $n \in \mathscr{A}_s$ for an $s \in \mathscr{S}_2$ having

$$p(b) > p_0(1 + 1/\log x). \tag{4.10}$$

Before proceeding further, observe that when $s \in \mathscr{S}_2$ satisfies (4.10), then

$$\frac{\sigma(b)}{b} > 1 + \frac{1}{p_0}.$$

Indeed, by the definition of p_0, there is some typical $s_0 \in (y, x]$ with the property that when we decompose $s_0 = a_0 b_0$ (say), the integer b_0 is composite and divisible by p_0. Since $a_0 p_0$ is a proper divisor of s_0, the integer $a_0 p_0$ is α-deficient. Since $s = ab$ is α-nondeficient,

$$\frac{\sigma(a_0)}{a_0}(1 + 1/p_0) < \alpha \leqslant \frac{\sigma(a)}{a}\frac{\sigma(b)}{b}.$$

By Lemma 4.4, $\sigma(a_0)/a_0 = \sigma(a)/a$, and so $\sigma(b)/b > 1 + 1/p_0$, as claimed.

Given $s \in \mathscr{S}_2$ satisfying (4.10), write $b = p_1 p_2 p_3 \cdots p_k$, numbered so that $p(b) = p_1 < p_2 < \cdots < p_k$. Then $k \leqslant 10 \log_2 x$, and so

$$1 + \frac{1}{p_0} < \frac{\sigma(b)}{b} \leqslant \left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right)^{10 \log_2 x}.$$

Hence,

$$\begin{aligned}\left(1 + \frac{1}{p_2}\right)^{10 \log_2 x} &> \left(1 + \frac{1}{p_0}\right)\left(1 + \frac{1}{p_1}\right)^{-1} \\ &\geqslant \left(1 + \frac{1}{p_0}\right)\left(1 + \frac{1}{p_0(1 + 1/\log x)}\right)^{-1} \\ &= 1 + \frac{1}{p_0 \log x(1 + 1/p_0 + 1/\log x)}.\end{aligned}$$

Taking logarithms,

$$\begin{aligned}\frac{10 \log_2 x}{p_2} &\geqslant (10 \log_2 x) \log\left(1 + \frac{1}{p_2}\right) \\ &\geqslant \log\left(1 + \frac{1}{p_0 \log x(1 + 1/p_0 + 1/\log x)}\right) \\ &\geqslant \frac{1}{2p_0 \log x(1 + 1/p_0 + 1/\log x)} \geqslant \frac{1}{3p_0 \log x}.\end{aligned}$$

Rearranging gives

$$p_2 \leqslant 30 p_0 \log x \log_2 x < p_0(\log x)^2.$$

Hence, $p_1, p_2 \in (p_0, p_0(\log x)^2)$.

We conclude that if $n \in \mathscr{A}_s$ for an $s \in \mathscr{S}_2$ satisfying (4.10), then n has at least two distinct prime divisors from $(p_0, p_0(\log x)^2)$. The number of such $n \leqslant x$ is at most

$$\frac{1}{2}x\Bigg(\sum_{\substack{p \text{ prime} \\ p_0 < p < p_0(\log x)^2}} \frac{1}{p}\Bigg)^2 \ll x\left(\frac{\log_2 x}{\log p_0}\right)^2 \ll \frac{x(\log_2 x)^2}{(\log x)^{1.8}} \ll \frac{x}{(\log x)^{3/2}}.$$

This completes the proof that (4.7) is $o((x/\log x)\log_2 x/\log_3 x)$ and also completes the proof of the theorem.

Remarks. There are two places where our estimates are not uniform in α. The first is when we apply Erdős's $o(x/\log x)$ upper bound for the count of primitive α-nondeficient numbers in $[1, x]$. That bound does not hold uniformly in α; indeed, every prime $p \leqslant x$ is primitive $(1 + 1/x)$-nondeficient, and there are asymptotically $x/\log x$ of these as $x \to \infty$. The second place non-uniformity is encountered is in our proof that the sum appearing in (4.8) is $o(1)$.

In both cases, slightly weaker uniform estimates are readily available. For example, the argument for Lemma 4.5 shows that the sum on a in (4.8) is bounded by an absolute constant. Making an essentially identical modification to Erdős's argument (crudely replacing "$c_7/b^{1/2}$" by "c_7" in [**15**, equation (30)]), Erdős's proof shows that the count of primitive α-nondeficient numbers in $[1, x]$ is $O(x/\log x)$, uniformly in α. Making use of these modified estimates in the proof presented above, we obtain that $E(\alpha; x) \ll (x/\log x)(\log_2 x/\log_3 x)$ for all x exceeding a certain *absolute* constant and all $\alpha \geqslant 1$. In other words, we recover the Elliott–Fainleib result quoted in the introduction as Theorem A.

§5. *Proof of Theorem 1.3.* Our starting point is the following observation of Fainleib, which is part of [**17**, Lemma 2].

LEMMA 5.1. *Let F and G be distribution functions satisfying $\int_{\mathbf{R}} |F(u) - G(u)|\, du < \infty$. Let $f(t) = \int_{\mathbf{R}} \mathrm{e}^{itu}\, dF(u)$ and $g(t) = \int_{\mathbf{R}} \mathrm{e}^{itu}\, dG(u)$ be the corresponding characteristic functions. For all real $T > 0$, we have*

$$\int_{\mathbf{R}} |F(u) - G(u)|^2\, du \leqslant \frac{4}{\pi T} + \frac{1}{2\pi} \int_{-T}^{T} \left| \frac{f(t) - g(t)}{t} \right|^2 dt.$$

Proof. For all real t, we have

$$f(t) - g(t) = \int_{\mathbf{R}} \mathrm{e}^{itu}\, d(F(u) - G(u)) = -it \int_{\mathbf{R}} (F(u) - G(u))\mathrm{e}^{itu}\, du.$$

So for $t \neq 0$,

$$\int_{\mathbf{R}} (F(u) - G(u))\mathrm{e}^{itu}\, du = i\,\frac{f(t) - g(t)}{t}.$$

Since $|F - G| \leqslant 2$ and $\int_{\mathbf{R}} |F(u) - G(u)|\, du < \infty$, we see that $\int_{\mathbf{R}} |F(u) - G(u)|^2\, du < \infty$. So by Parseval's identity,

$$\begin{aligned} \int_{\mathbf{R}} |F(u) - G(u)|^2\, du &= \frac{1}{2\pi} \int_{\mathbf{R}} \left| \frac{f(t) - g(t)}{t} \right|^2 dt \\ &\leqslant \frac{1}{2\pi} \int_{(-\infty, T) \cup (T, \infty)} \frac{4}{t^2}\, dt + \frac{1}{2\pi} \int_{-T}^{T} \left| \frac{f(t) - g(t)}{t} \right|^2 dt \\ &= \frac{4}{\pi T} + \frac{1}{2\pi} \int_{-T}^{T} \left| \frac{f(t) - g(t)}{t} \right|^2 dt. \qquad \square \end{aligned}$$

We now introduce the relevant distribution functions. For each natural number N, let

$$F_N(u) := \frac{1}{N} \# \left\{ n \leqslant N : \log \frac{\sigma(n)}{n} \leqslant u \right\}.$$

Let $G(u)$ denote the natural density of those n with $\sigma(n) \leqslant e^u n$. Note that G is well defined by Davenport's theorem, and in fact

$$G(u) = 1 - D(e^u).$$

By the definition of weak convergence, we see that F_N converges weakly to G as $N \to \infty$. We let $f_N(t)$ be the characteristic function of F_N and we let $g(t)$ be the characteristic function of G. We will need the following convenient expression for $g(t)$.

LEMMA 5.2. *Let t be a real number. Let h be the arithmetic function defined by*

$$\left(\frac{\sigma(n)}{n}\right)^{it} = \sum_{d|n} h(d). \tag{5.1}$$

(Of course, h depends on t, but we suppress this in our notation.) Then

$$g(t) = \sum_{d=1}^{\infty} \frac{h(d)}{d}.$$

Proof. Since $F_n \Rightarrow G$, we find that $g(t) = \lim_{N\to\infty} f_N(t) = \lim_{N\to\infty}(1/N) \sum_{n=1}^{N}(\sigma(n)/n)^{it}$. In other words, $g(t)$ is the mean value of $(\sigma(n)/n)^{it}$. Referring back to Lemma 2.7, we see that the claimed expression for $g(t)$ will follow if we show that $\sum_{d=1}^{\infty} h(d)/d$ converges absolutely. Now

$$\sum_{d=1}^{\infty} \frac{|h(d)|}{d} = \prod_{p \text{ prime}} \left(1 + \frac{|h(p)|}{p} + \frac{|h(p^2)|}{p^2} + \cdots\right) \leqslant \exp\Bigg(\sum_{\substack{p \text{ prime} \\ k\geqslant 1}} \frac{|h(p^k)|}{p^k}\Bigg). \tag{5.2}$$

Each term $h(p^k) = (\sigma(p^k)/p^k)^{it} - (\sigma(p^{k-1})/p^{k-1})^{it}$, and thus $|h(p^k)| \leqslant 2$ trivially. Hence, the terms corresponding to $k \geqslant 1$ make a bounded contribution to the final sum in (5.2), and so it is enough to prove that $\sum_{p \text{ prime}} |h(p)|/p < \infty$. To this end, we observe that when $p > |t|$, we have $|it \log(1 + 1/p)| \leqslant |t|/p \leqslant 1$; so from the Maclaurin expansion of $\exp(\cdot)$,

$$|h(p)| = \left|\exp\left(it \log\left(1 + \frac{1}{p}\right)\right) - 1\right| \ll \frac{|t|}{p}. \tag{5.3}$$

This immediately implies convergence of $\sum_p |h(p)|/p$. □

Assume that $x \geqslant 3$ is given. In what follows, the letter F, sans subscript, denotes the distribution function $F_{\lfloor x \rfloor}$. We use $f = f_{\lfloor x \rfloor}$ to denote the characteristic function of F.

LEMMA 5.3. *Suppose that x exceeds a suitable absolute constant.*

(i) *If $|t| \leqslant \frac{1}{4}$, then $|f(t) - g(t)| \ll |t|$.*
(ii) *For all real t,*

$$f(t) - g(t) \ll \exp\left(-\frac{1}{3}\frac{\log x}{\log(|t| + 3)}\right) \cdot (\log (|t| + 3))^3.$$

(iii) *For all real t with $|t| > (\log x)^2$,*

$$f(t) - g(t) \ll \exp\left(-\frac{1}{2}\frac{\log x \log_2 x}{\log |t|}\right) \cdot \exp\left(O((\log x)^{1/2})\right) \cdot (\log |t|)^{O(1)}.$$

All implied constants are absolute.

Before proving Lemma 5.3, we record a useful estimate extracted from [25] (see equation (2.4) there). We use the notation li for the logarithmic integral, so that $\mathrm{li}(y) = \int_2^y (1/\log t)\, dt$.

LEMMA 5.4. *Suppose that $0 < \eta < 1$. If $y^{1-\eta} \geqslant 2$, then*

$$\sum_{\substack{p \text{ prime} \\ p \leqslant y}} p^{-\eta} = \mathrm{li}(y^{1-\eta})\left(1 + O\left(\frac{1}{\log y}\right)\right) + O\left(\log \frac{1}{1-\eta}\right).$$

Lemma 5.4 can be obtained from partial summation and the prime number theorem with the classical (de la Vallée–Poussin) error term.

Proof. The estimate (i) is implicitly contained in Elliott's monograph. Indeed, the second half of [8, Lemma 5.7, p. 203] asserts that $f_N(t) = 1 + O(|t|)$ for $|t| \leqslant 1/4$, uniformly in N. Since $f(t) = f_{\lfloor x \rfloor}(t)$ while $g(t) = \lim_{N\to\infty} f_N(t)$, we see that both $f(t)$ and $g(t)$ are also $1 + O(|t|)$ in this range of t. Subtracting gives (i).

Part (ii) is essentially established in Fainleib's paper [17, proof of Lemma 1]. We give the argument in detail, since we will need it when proving (iii). Define the arithmetic function h so that (5.1) holds. Then

$$f(t) = \frac{1}{\lfloor x \rfloor}\sum_{n \leqslant x}\left(\frac{\sigma(n)}{n}\right)^{it} = \frac{1}{\lfloor x \rfloor}\sum_{d \leqslant x} h(d)\left\lfloor \frac{\lfloor x \rfloor}{d}\right\rfloor = \sum_{d \leqslant x}\frac{h(d)}{d} + O\left(\frac{1}{x}\sum_{d \leqslant x}|h(d)|\right).$$

From Lemma 5.2, we know that $g(t) = \sum_{d=1}^{\infty} h(d)/d$. Thus,

$$f(t) = g(t) + O\left(\sum_{d > x}\frac{|h(d)|}{d} + \frac{1}{x}\sum_{d \leqslant x}|h(d)|\right).$$

Let η be a real parameter whose precise value will be chosen shortly; for now, we assume only that $\frac{2}{3} < \eta < 1$. Observe that $x^{\eta-1}d^{-\eta} = (x/d)^{\eta}x^{-1} \geqslant x^{-1}$

when $d \leqslant x$, while $x^{\eta-1}d^{-\eta} = (x/d)^{\eta-1}d^{-1} \geqslant d^{-1}$ when $d > x$. Hence,

$$\begin{aligned} f(t) - g(t) &\ll \sum_{d>x} \frac{|h(d)|}{d} + \frac{1}{x}\sum_{d\leqslant x} |h(d)| \leqslant x^{\eta-1}\sum_{d=1}^{\infty}\frac{|h(d)|}{d^\eta} \\ &= x^{\eta-1}\prod_{p \text{ prime}}\left(1 + \frac{|h(p)|}{p^\eta} + \frac{|h(p^2)|}{p^{2\eta}} + \cdots\right) \\ &\leqslant x^{\eta-1}\exp\Bigg(\sum_{\substack{p \text{ prime}\\ k\geqslant 1}} \frac{|h(p^k)|}{p^{k\eta}}\Bigg). \end{aligned}$$

As noted in the proof of Lemma 5.2, each term $|h(p^k)| \leqslant 2$. So using that $\eta > \frac{2}{3}$, we see that $\sum_{p \text{ prime}, k\geqslant 2} |h(p^k)|/p^{k\eta}$ is absolutely bounded. Thus,

$$f(t) - g(t) \ll x^{\eta-1}\exp\Bigg(\sum_{p \text{ prime}} \frac{|h(p)|}{p^\eta}\Bigg). \tag{5.4}$$

We now choose

$$\eta = 1 - \frac{1}{3\log(|t|+3)}.$$

(Note that $\frac{2}{3} < \eta < 1$, as required.) Then

$$x^{\eta-1} = \exp\left(-\frac{1}{3}\frac{\log x}{\log(|t|+3)}\right). \tag{5.5}$$

We split the sum on p in (5.4) at $|t|+3$. For $p \leqslant |t|+3$, we have

$$\frac{|h(p)|}{p^\eta} \leqslant \frac{2}{p}\cdot p^{1-\eta} \leqslant \frac{2}{p}(|t|+3)^{1-\eta} = 2\exp(1/3)\cdot\frac{1}{p} < \frac{3}{p},$$

and thus

$$\sum_{\substack{p \text{ prime}\\ p\leqslant |t|+3}} \frac{|h(p)|}{p^\eta} < 3\sum_{\substack{p \text{ prime}\\ p\leqslant |t|+3}} \frac{1}{p} = 3\log_2(|t|+3) + O(1). \tag{5.6}$$

For $p > |t|+3$, we know from (5.3) that $|h(p)| \ll |t|/p$. Hence,

$$\sum_{\substack{p \text{ prime}\\ p> |t|+3}} \frac{|h(p)|}{p^\eta} \ll |t|\sum_{\substack{p \text{ prime}\\ p> |t|+3}} \frac{1}{p^{\eta+1}} \leqslant |t|\sum_{n>|t|+3}\frac{1}{n^{\eta+1}} \ll (|t|+3)^{1-\eta} \ll 1. \tag{5.7}$$

Combining (5.4)–(5.7) gives the estimate (ii).

Part (iii) is very similar, except that now we apply (5.4) with

$$\eta = 1 - \frac{1}{2}\frac{\log_2 x}{\log|t|}.$$

(Since $|t| > (\log x)^2$, we have $\frac{3}{4} < \eta < 1$.) Here

$$x^{\eta-1} = \exp\left(-\frac{1}{2}\frac{\log x \log_2 x}{\log |t|}\right). \tag{5.8}$$

This time, we split the sum on p appearing in (5.4) at $|t|$. Applying Lemma 5.4, and noting that $|t|^{1-\eta} = (\log x)^{1/2}$, we find that

$$\begin{aligned}\sum_{\substack{p \text{ prime}\\ p\leqslant |t|}} \frac{|h(p)|}{p^\eta} &\leqslant 2 \sum_{\substack{p \text{ prime}\\ p\leqslant |t|}} \frac{1}{p^\eta}\\ &= 2\cdot \mathrm{li}((\log x)^{1/2})(1+O(1/\log |t|)) + O(\log_2 |t|)\\ &\ll (\log x)^{1/2}/\log_2 x + \log_2 |t|. \end{aligned} \tag{5.9}$$

Using once more that $|h(p)| \ll |t|/p$ for $p > |t|$, we find that

$$\sum_{\substack{p \text{ prime}\\ p>|t|}} \frac{|h(p)|}{p^\eta} \ll |t| \sum_{\substack{p \text{ prime}\\ p>|t|}} \frac{1}{p^{\eta+1}} \leqslant |t| \sum_{n>|t|} \frac{1}{n^{\eta+1}} \ll |t|^{1-\eta} = (\log x)^{1/2}. \tag{5.10}$$

The estimate (iii) now follows from assembling (5.4) and (5.8)–(5.10). □

The following estimate, which shows that $D(\alpha)$ decays to 0 extremely rapidly as $\alpha \to \infty$, is due to Erdős [14, Theorem 1]. While this will more than suffice for our purposes, we note that more precise results have recently been obtained by Weingartner [31, 32].

LEMMA 5.5. *As $\alpha \to \infty$, we have*

$$D(\alpha) = \exp(-\exp((\mathrm{e}^{-\gamma} + o(1))\alpha)).$$

As before, γ denotes the Euler–Mascheroni constant.

Proof of Theorem 1.3. Our strategy will be to first bound the integral of $|E(\alpha; x)|^2$ over $\alpha \in [1, 2\log_2 x]$. We will then use Lemma 5.5 to show that the integral taken over the remaining range $\alpha > 2\log_2 x$ is negligible. In fact, we will see that this range of α makes a contribution that is $o(1)$, as $x \to \infty$.

With an eye towards applying Lemma 5.1, we start by relating $E(\alpha; x)$ to the difference $F - G$. For all $\alpha \geqslant 1$,

$$F(\log \alpha) - G(\log \alpha) = \frac{1}{\lfloor x \rfloor}\left(\left(\sum_{\substack{n\leqslant x\\ \sigma(n)\leqslant \alpha n}} 1\right) - G(\log \alpha)\lfloor x \rfloor\right).$$

Moreover, using that $G(\log\alpha) = 1 - D(\alpha)$,

$$\begin{aligned}&\Bigg(\sum_{\substack{n\leqslant x\\ \sigma(n)\leqslant \alpha n}} 1\Bigg) - G(\log\alpha)\lfloor x\rfloor \\ &\quad= \left(\lfloor x\rfloor - A(\alpha;x) + \#\left\{n\leqslant x : \frac{\sigma(n)}{n} = \alpha\right\}\right) - (1-D(\alpha))\lfloor x\rfloor \\ &\quad= -D(\alpha)(x-\lfloor x\rfloor) + \#\left\{n\leqslant x : \frac{\sigma(n)}{n} = \alpha\right\} - E(\alpha;x).\end{aligned}$$

Rearranging, and using Wirsing's upper bound of $x^{W/\log_2 x}$ for the number of $n\leqslant x$ with $\sigma(n)/n = \alpha$, we find that

$$E(\alpha;x) \ll x^{W/\log_2 x} + x|F(\log\alpha) - G(\log\alpha)|.$$

Hence,

$$\begin{aligned}&\int_1^{2\log_2 x} |E(\alpha;x)|^2\,d\alpha \\ &\quad\ll \int_1^{2\log_2 x} (x^{2W/\log_2 x} + x^2|F(\log\alpha) - G(\log\alpha)|^2)\,d\alpha \\ &\quad\ll (\log_2 x)\left(x^{2W/\log_2 x} + x^2\int_1^{2\log_2 x} |F(\log\alpha) - G(\log\alpha)|^2\alpha^{-1}\,d\alpha\right) \\ &\quad\leqslant x^{2W/\log_2 x}\cdot \log_2 x + x^2\log_2 x\int_{\mathbf{R}} |F(u)-G(u)|^2\,du. \qquad (5.11)\end{aligned}$$

We would now like to apply Lemma 5.1, but must first check that $F - G \in L^1(\mathbf{R})$. We start by observing that for $u < 0$, we have $F(u) - G(u) = 0$. Since $\sigma(n)/n$ assumes only finitely many values for $n \leqslant x$, it follows that $F(u) = 1$ for all large enough positive values of u. Hence, $F(u) - G(u) = 1 - G(u) = D(\mathrm{e}^u)$ for large positive u. Since $D(\mathrm{e}^u)$ decays extremely rapidly to 0 by Lemma 5.5, we conclude that $\int_{\mathbf{R}} |F(u) - G(u)|^2\,du < \infty$, as desired.

Now let

$$T := \exp\left(\sqrt{\tfrac{1}{2}\log x\log_2 x}\right).$$

According to Lemma 5.1,

$$\int_{\mathbf{R}} |F(u)-G(u)|^2\,du \leqslant \frac{4}{\pi T} + \frac{1}{2\pi}\int_{-T}^{T}\left|\frac{f(t)-g(t)}{t}\right|^2 dt. \qquad (5.12)$$

We break the integral appearing on the right-hand side of (5.12) into three pieces. First, we consider those values of t with $|t| \leqslant 1/T$. By Lemma 5.3(i), this range of t contributes $O(1/T)$ to the integral. Next, we consider those t with $1/T < |t| \leqslant (\log x)^2$. For these t, Lemma 5.3(ii) gives

$$f(t) - g(t) \ll \exp\left(-\frac{1}{7}\frac{\log x}{\log_2 x}\right).$$

So the integral of $|f(t) - g(t)|^2/t^2$ over this range of t is

$$\ll (\log x)^2 \cdot T^2 \cdot \exp\left(-\frac{2}{7}\frac{\log x}{\log_2 x}\right) \ll \exp\left(-\frac{1}{4}\frac{\log x}{\log_2 x}\right).$$

For $(\log x)^2 < |t| \leqslant T$, we use Lemma 5.3(iii), which shows that

$$f(t) - g(t) \ll \exp(-\sqrt{(1/2 + o(1)) \log x \log_2 x}).$$

Thus, the integral of $|f(t) - g(t)|^2/t^2$ over this range of t is

$$\begin{aligned} &\ll T(\log x)^{-4} \cdot \exp(-\sqrt{(2 + o(1)) \log x \log_2 x}) \\ &\ll \exp(-\sqrt{(1/2 + o(1)) \log x \log_2 x}). \end{aligned}$$

Since $T^{-1} = \exp(-\sqrt{\frac{1}{2} \log x \log_2 x})$, we find after collecting all of our estimates that the right-hand side of (5.12) is at most $\exp(-\sqrt{(1/2 + o(1)) \log x \log_2 x})$, as $x \to \infty$. Putting this back into (5.11), we see that as $x \to \infty$,

$$\int_1^{2\log_2 x} |E(\alpha; x)|^2 \, d\alpha \leqslant x^2 \exp(-\sqrt{(1/2 + o(1)) \log x \log_2 x}).$$

Finally, we show that $\int_{2\log_2 x}^{\infty} |E(\alpha; x)|^2 \, d\alpha = o(1)$, as $x \to \infty$. For sufficiently large x, Gronwall's result gives that $A(2\log_2 x; x) = 0$. So for $\alpha > 2\log_2 x$, we have $|E(\alpha; x)|^2 = x^2 D(\alpha)^2$. Since $\mathrm{e}^{-\gamma} > 0.56$, Lemma 5.5 guarantees that for each $j = 1, 2, 3, \ldots,$

$$\begin{aligned} \int_{2^j \log_2 x}^{2^{j+1}\log_2 x} D(\alpha)^2 \, d\alpha &\leqslant 2^j \log_2 x \cdot \exp(-2\exp(2^j \cdot 0.56 \log_2 x)) \\ &= 2^j \log_2 x \cdot \exp(-2(\log x)^{1.12 \cdot 2^{j-1}}). \end{aligned}$$

(Here we again assume that x is sufficiently large.) With

$$U_j := 2^j \log_2 x \cdot \exp(-2(\log x)^{1.12 \cdot 2^{j-1}}),$$

it is straightforward to check that $U_{j+1} \leqslant \frac{1}{2} U_j$ for all j. Thus,

$$\begin{aligned} \int_{2\log_2 x}^{\infty} |E(\alpha; x)|^2 \, d\alpha = x^2 \int_{2\log_2 x}^{\infty} D(\alpha)^2 \, d\alpha \\ \leqslant x^2 \sum_{j=1}^{\infty} U_j \leqslant 2x^2 \cdot U_1 \\ = 4x^2 \log_2 x \cdot \exp(-2(\log x)^{1.12}), \end{aligned}$$

which is indeed $o(1)$ as $x \to \infty$. This completes the proof of the theorem. □

§6. *A concluding challenge.* Let f be a positive-valued multiplicative function. We say that the natural number n is (f, α)-*abundant* if $f(n) \geqslant \alpha$; our work here corresponds to the choice $f(n) = \sigma(n)/n$. Under certain technical conditions on f, Erdős [**11, 12, 13**] showed that the (f, α)-abundant numbers possess an asymptotic density $D_f(\alpha)$, where $D_f(\alpha) \to 1$ as $\alpha \to 0$ and $D_f(\alpha) \to 0$ as $\alpha \to \infty$. (This is the sufficiency half of the Erdős–Wintner theorem [**16**].) It would be interesting to adapt the methods of this paper to study the analogous questions about error terms in this general setting. This would seem to require a theory of primitive (f, α)-abundant numbers robust enough to produce generalizations of both Lemma 2.1 and Proposition 2.3.

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