Preface

These are notes from a 5-week course that I (P.P.) taught as part of the 2019 Ross/Asia Mathematics Program. The course and camp were held from July 7 through August 9 at Jiangsu Aviation Technical College in Zhenjiang, Jiangsu, China. Clientele ranged widely in age and mathematical experience; most participants were high school students, but some were undergraduates and others were mathematics graduate students.

The course was nontraditional in format. In lieu of lectures, participants received 15 problem sets (or "Steps") over the course of the summer, one every two to three days. Class meetings (held each weekday, and sometimes also on weekends) were entirely devoted to presentations of solutions, sometimes by students and sometimes by the instructor. Participants were expected to have a strong background in mathematical problem solving (e.g., from training for mathematics contests) but not assumed to possess advanced subject-matter knowledge. As such, most of the solutions require nothing more than elementary number theory and a solid grasp of calculus. The chief exception is the proof of Dirichlet's theorem on primes in progressions for a general modulus, where we use (without proof) certain facts from complex variables. Even there, someone familiar with the real-variables side of things will find the necessary results easy enough to swallow.

It is not at all obvious to the uninitiated that analysis has something of value to offer arithmetic. I attempted in this course to marshal the most convincing examples available for this surprising thesis. This explains the somewhat atypical emphasis throughout on concrete, number-theoretic problems, in contrast to a systematic development of analytic tools. Our primary themes are the value-distribution of arithmetic functions (e.g., Hardy and Ramanujan's result on the typical number of prime factors of an integer and Erdős's multiplication table theorem), the distribution of prime numbers (Chebyshev's results, Dirichlet's theorem, Brun's theorem on twin primes), and the distribution of squares and nonsquares modulo p (e.g., Vinogradov's upper bound on the least positive nonsquare mod p). Of course, in 5 weeks one can only cover so much; somewhat regrettably, these notes do not include a proof of the Prime Number Theorem.

Apart from the addition of a “problem track” on the values of $\zeta(s)$ at positive even integers, the problem sets are mostly unchanged from what students received in Summer 2019. What *is* new are the solution sets, which have been prepared by myself and Akash Singha Roy (a 2019 Ross Program counselor and enthusiastic participant in the original course). Our intent with the solution sets was to provide enough detail that novices will find the text useful for self-study. Several of the problems come attached to remarks indicating directions for interested students to explore further.

Akash and I would like to conclude by thanking all of the student participants in the original course, as well as the “powers that be” behind the Ross program: Tim All, Jim Fowler, Dan Shapiro, and Jerry Xiao.

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Notation

Most of our notation and conventions will already be familiar to anyone who has taken a course in elementary number theory. Here are some possible exceptions: When we write “log”, we always intend the natural logarithm. The set of positive integers is denoted $\mathbb{Z}^+$. For us, the letter p always denotes a prime, whether or not this is mentioned explicitly. We use $\mathbb{Z}_m$ for the integers mod m and write $\mathbb{U}_m$ for the group of units mod m. The residue class of the integer a, modulo m, is denoted “$a \bmod m$”. Finally, sums over integers are to be understood as taken only over positive integers, unless explicitly indicated otherwise. For example, “$\sum_{n \le x}$” means a sum over positive integers $n \le x$.

Contents