Errata to: Finiteness theorems for perfect numbers and their kin

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In the title article, the author leaves to the reader the task of showing that the abundancy function $h\colon \mathscr{S} \to \widehat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is continuous. As noted by Richard Stone, this claim is in fact false! For example, if $N_k = \prod_{p>k} p$, then $N_k \to 1$ in the supernatural topology; on the other hand, each $h(N_k) = \infty$ (since the sum of the reciprocals of the primes diverges) while $h(1) = 1$.

The following result should be substituted in place of the erroneous claim above. For each natural number B, let $\mathscr{S}^B$ denote the set of supernatural numbers N whose support has size bounded by B.

Proposition. *For each B, the set $\mathscr{S}^B$ is a closed subset of $\mathscr{S}$. Moreover, the restriction of h to $\mathscr{S}^B$ defines a continuous function to $\widehat{\mathbf{R}}$.*

From the first half of the proposition, we see that if N_i is any convergent sequence in $\mathscr{S}^B$ with limit N, then $N \in \mathscr{S}^B$; we then deduce from the second half that $h(N_i) \to h(N)$. These results suffice for all of the applications given in the paper, and allow their proofs to remain essentially unchanged.

Before we prove the proposition, we introduce a convenient decomposition of the function h. Suppose that $y > 1$. If M is a supernatural number, say $M = \prod_p p^{e_p}$, we set

$$h_y(M) = \prod_{p<y} h(p^{e_p}), \quad \text{and} \quad h^y(M) = \prod_{p\geq y} h(p^{e_p}).$$

Then for every $M \in \mathscr{S}$, we have

$$h(M) = h_y(M)h^y(M).$$

Moreover, if $M \in \mathscr{S}^B$, then

$$1 \leq h^y(M) = \prod_{p\geq y} h(p^{e_p}) \leq \prod_{\substack{p\geq y\\ p\mid M}} \frac{p}{p-1} \leq \left(1 + \frac{1}{y-1}\right)^B. \tag{1}$$

In particular, taking $y = 2$, we see that h is bounded on $\mathscr{S}^B$.

Lemma. *For each choice of y, the function $h_y\colon \mathscr{S} \to \widehat{\mathbf{R}}$ is continuous.*

For all of the proofs given below, we take advantage of the fact that $\mathscr{S}$ is metrizable, being a countable product of metrizable spaces. (To see the last claim, first observe that $\widehat{\mathbf{R}}$ is homeomorphic to the circle, and so $\widehat{\mathbf{N}}$ can be thought of as a subset of the circle.) So for instance, in proofs of continuity we may use the "convergent sequence" characterization.

Proof of the lemma. If $N_i \to N$, then $v_p(N_i) \to v_p(N)$ for each prime p. We may deduce from this that for every prime p, we have $h(p^{v_p(N_i)}) \to h(p^{v_p(N)})$. Indeed, this last assertion is clear if $v_p(N) < \infty$, since in that case $v_p(N_i) = v_p(N)$ for all large i. On the other hand, if $v_p(N) = \infty$, then $v_p(N_i) \to \infty$, and so here too we have

$$h(p^{v_p(N_i)}) = \sum_{0 \le j \le v_p(N_i)} p^{-j} \to \frac{p}{p-1} = h(p^{v_p(N)}).$$

Since there are *finitely many* primes $p < y$, we conclude that

$$h_y(N_i) = \prod_{p<y} h(p^{v_p(N_i)}) \to \prod_{p<y} h(p^{v_p(N)}) = h_y(N).$$

This completes the proof. □

Proof of the proposition. First we prove that $\mathscr{S}^B$ is closed. Each element N of the closure of $\mathscr{S}^B$ is the limit of a sequence of points $N_i \in \mathscr{S}^B$. Supposing for the sake of contradiction that N is supported on more than B primes, choose primes $p_1, \dots, p_{B+1}$ dividing N. Since $v_p(N_i) \to v_p(N)$ for each p, it must be that $p_1 \cdots p_{B+1} \mid N_i$ for all large i, contradicting that $N_i \in \mathscr{S}^B$. Next, we show continuity of $h|_{\mathscr{S}^B}$. Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of points of $\mathscr{S}^B$ converging to $N \in \mathscr{S}^B$. Fix any $\epsilon > 0$. For each y, we can write

$$h(N_i)/h(N) = \frac{h_y(N_i)}{h_y(N)} \cdot \frac{h^y(N_i)}{h^y(N)}. \tag{2}$$

Applying (1) with $M = N$ and $M = N_i$, we see that the second right-hand factor belongs to the interval $(1-\epsilon, 1+\epsilon)$ for large enough y, depending only on B and ϵ. Now fix such a y. By the lemma, the first right-hand factor in (2) tends to 1 as $i \to \infty$. (We use here that $h_y(N) < \infty$.) Thus,

$$1 - \epsilon \le \liminf_{i\to\infty} h(N_i)/h(N) \le \limsup_{i\to\infty} h(N_i)/h(N) \le 1 + \epsilon.$$

But $\epsilon > 0$ was arbitrary. Hence, $h(N_i)/h(N) \to 1$, and $h(N_i) \to h(N)$. This completes the proof of continuity. □