Thoughts on the order of a mod p

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Komal Agrawal, UGA

Zeb Engberg, Wasatch Academy

Out of chaos. . .

Let a be an integer, $a \neq 0, \pm 1$. For each integer m relatively prime to a, we define

$$\ell_a(m) = \text{multiplicative order of } a \bmod m.$$

In other words, $\ell_a(m)$ is the least positive integer ℓ for which

$$a^\ell \equiv 1 \pmod{m}.$$

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We are interested in understanding the distribution of $\ell_a(p)$ as p varies, either with a fixed, or a belonging to a finite set.

There is nothing like looking, if you want to find something. – J.R.R. Tolkien

Fix $a = 2$ and write $\ell(p)$ rather than $\ell_2(p)$.

There are 78498 primes $p \leq 10^6$. And $\ell(p)$ is defined for 78497 of these.

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For 29341 of these, have $\ell(p) = p - 1$.
For 22092 of these, have $\ell(p) = (p-1)/2$.
For 5233 of these, have $\ell(p) = (p-1)/3$.
For 3655 of these, have $\ell(p) = (p-1)/4$.
For 1477 of these, have $\ell(p) = (p-1)/5$.

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These cases account for about 79% of the primes $p \le 10^6$.

Artin's primitive root conjecture

Conjecture (E. Artin, 1927)

Fix $a \in \mathbb{Z}$, not a square, and not ± 1. There are infinitely many primes p for which $\ell_a(p) = p - 1$. In fact, the number of primes $p \leq x$ with $\ell(p) = p - 1$ is

$$\sim C(a)\pi(x),$$

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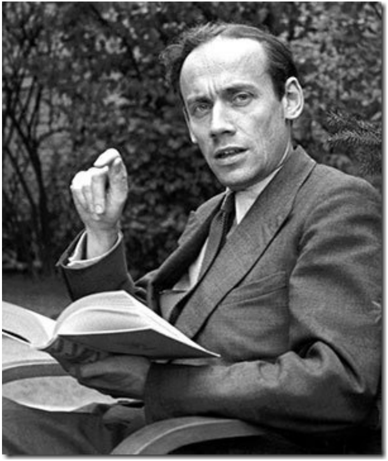
where $C(a)$ is an explicitly described positive constant.

When $a = 2$, he predicts

$$C(2) = \prod_p \left(1 - \frac{1}{p(p-1)}\right)$$
$$= 0.3739558...$$

Of the 78498 primes $p \leq 10^6$, 29341 have 2 as a primitive root: $29341/78498 = 0.37378\ldots$.



Emil Artin

So close and yet so far

Hooley (1967): Artin's conjecture is correct ... assuming GRH!

Hooley's work implies that (on GRH) $\ell(p)$ is usually fairly close to $p-1$. If $\xi(x) \to \infty$ as $x \to \infty$, no matter how slowly, then almost all primes p satisfy

$$\frac{p-1}{\ell(p)} < \xi(p).$$

"Almost all": Asymptotically 100%.

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First half of this talk: What can we say unconditionally?

Theorem (Heath-Brown, Gupta–Murty)

At least one of $2, 3, 5$ *is a primitive root for infinitely many primes* p. *That is, there is some* $a \in \{2, 3, 5\}$ *such that*

$$\ell_a(p) = p - 1$$

for infinitely many primes p. *Moreover,* $2, 3, 5$ *can be replaced with any three distinct primes.*

Their proofs give: $\gg x/(\log x)^2$ such primes $p \leq x$.

Question

What kind of lower bound on $\ell_a(p)$ *can be shown to hold for a positive proportion of primes* p? *Or for almost all primes* p?

Ram Murty

Theorem (Hooley)

Fix $\epsilon > 0$. Fix $a \notin \{0, \pm 1\}$. For almost all primes p,

$$\ell_a(p) > p^{1/2-\epsilon}.$$

Proof.

We give the proof when $a = 2$.

Suppose $p \le x$ and $\ell_2(p) \le p^{1/2-\epsilon} \le x^{1/2-\epsilon} := X$. Then

$$p \mid 2^{\ell_2(p)} - 1 \mid (2^1 - 1)(2^2 - 1) \cdots (2^{\lfloor X \rfloor} - 1).$$

The product is $< 2^{X^2}$ and so has $< X^2 = x^{1-2\epsilon}$ prime factors. And X^2 is asymptotically 0% of $\pi(x)$, as $x \to \infty$.

This observation was extended by Matthews.

Theorem (Matthews)

Fix $\epsilon > 0$ and fix a positive integer k.
Suppose $a_1, \ldots, a_k$ are multiplicatively independent nonzero integers. Then for almost all primes p, the order of the subgroup mod p generated by $a_1, \ldots, a_k$ is at least

$$p^{\frac{k}{k+1}-\epsilon}.$$

The proof is similar: With $a_1, \ldots, a_k$ as above, one shows there are few primes "dividing" the rational numbers

$$a_1^{n_1} \cdots a_k^{n_k} - 1,$$

for nonzero tuples $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ of small height, meaning $\max|n_i| \leq n^{(1-\epsilon)/(k+1)}$.

Theorem (Kurlberg–Pomerance)

For each fixed $a \notin \{0, \pm 1\}$, Kurlberg–Pomerance showed that a positive proportion of primes p satisfy

$$\ell_a(p) > p^{0.677}.$$

Here is their simple proof: By a result of Baker–Harman, a positive proportion of p are such that $p-1$ has a prime factor $> p^{0.677}$. If $\ell_a(p)$ is divisible by that prime, then $\ell_a(p) > p^{0.677}$ also. If not, then $\ell_a(p) < (p-1)/p^{0.677} < p^{0.323}$, which is very rare (0% of primes, by Hooley).

Almost all?

Hooley's exponent $\frac{1}{2}$ has resisted improvement for more than 50 years.

The "record" result in this direction is due to Erdős and Murty and replaces $\frac{1}{2} - \epsilon$ with $\frac{1}{2} + \epsilon(p)$: *If $\epsilon(p)$ is any function tending to 0 as $p \to \infty$, then*

$$\ell_a(p) > p^{\frac{1}{2}+\epsilon(p)}$$

for almost all primes p.

Komal and I showed that we can break the "$\frac{1}{2}$-barrier" for a slightly different question.

Theorem (Agrawal and P., 2020)

Fix $\epsilon > 0$. For almost all primes p, there is an $a \in \{2, 3, 6, 12, 18\}$ with

$$\ell_a(p) > p^{8/15-\epsilon}.$$

Note that $8/15 = 1/2 + 1/30$.

One can replace 2, 3, 6, 12, 18 with a, b, ab, a^2b, ab^2 for multiplicatively independent nonzero integers a, b.

Our proof uses the results of Hooley and Matthews, along with the following undergraduate-level exercise, applied to the multiplicative group mod p.

Proposition

Let G be a cyclic group of order M whose order is divisible by p but not p^2, with generator g. Let $\log_g \colon G \to \mathbb{Z}/M\mathbb{Z}$ be the "discrete log" base g. Then for each $a \in G$,

$$p \mid \text{order of } a \iff p \nmid \log_g(a).$$

To prove the $8/15$ theorem, we look at the prime factorization of the product

$$\ell_2(p)\ell_3(p)\ell_6(p)\ell_{12}(p)\ell_{18}(p).$$

Let $L = \mathrm{lcm}[\ell_2(p), \ell_3(p)]$.

Observe that each of $2, 3, 6, 12, 18$ has order dividing L. Hence, every prime dividing our 5-fold product divides L.

Using the elementary group theoretic fact described above, we show that "typically" a prime dividing L divides at least four of the five terms in the product.

What is it we really show about $L = \mathrm{lcm}[\ell_2(p), \ell_3(p)]$?

Let $F = \lfloor \log\log p \rfloor !$. We show that for almost all primes p,

$$L^4 \mid F\ell_2(p)\ell_3(p)\ell_6(p)\ell_{12}(p)\ell_{18}(p).$$

Note that F is small: in particular, $F < p^{\epsilon}$.

Hence,

$$\ell_2(p)\ell_3(p)\ell_6(p)\ell_{12}(p)\ell_{18}(p) > L^4 p^{-\epsilon}.$$

The result of Matthews gives $L > p^{2/3-\epsilon}$, almost always.

Hence,

$$\ell_2(p)\ell_3(p)\ell_6(p)\ell_{12}(p)\ell_{18}(p) > p^{8/3-5\epsilon}.$$

Now take fifth roots and view LHS as a geometric mean.

A remark

One can get exponents larger than $8/15$ but working with larger sets.

Theorem

For each $\epsilon > 0$, there is a finite set $\mathcal{A}$ such that, for almost all primes p, some $a \in \mathcal{A}$ satisfies

$$\ell_a(p) > p^{1-\epsilon}.$$

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Theorem

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Consequently (Pigeonhole Principle), there is a (fixed) $a \in \mathcal{A}$ such that

$$\ell_a(p) > p^{1-\epsilon}$$

on a set of primes p of upper density at least $1/|\mathcal{A}| > 0$.

For example, there is a positive integer a such that, on a set of primes p of positive upper density,

$$\ell_a(p) > p^{0.999}.$$

One can also get this going for composite numbers.

Let $\ell_a^*(n)$ be the length of the period of the sequence $a, a^2, a^3, \ldots$ modulo n. Then for almost all n, there is an $a \in \{2, 3, 6, 12, 18\}$ with

$$\ell_a^*(n) > n^{8/15-\epsilon}.$$

Again this goes through for a, b, ab, a^2b, ab^2 if a, b are multiplicatively independent.

One can also incorporate the $+\epsilon(p)$ improvement of Erdős–Murty. As an example, if $\epsilon(p) \to 0$, then for almost all primes p, there is an $a \in \{2, 3, 6, 12, 18\}$ with

$$\ell_a(p) > p^{8/15+\epsilon(p)}.$$

Part II: Mersenne numbers

We would like to understand arithmetic properties of Mersenne numbers $2^n - 1$.

As an example of a natural question, it would be good to understand the average order of the arithmetic function

$$\omega(2^n - 1) = \sum_{p \mid 2^n - 1} 1.$$

We have only very weak results on this problem: with $\ell(p) = \ell_2(p)$, it comes down to estimating $\sum_{p \leq x} \frac{1}{\ell(p)}$, which appears very difficult.

The situation gets easier if we replace the summand 1 with a weight that dampens the sensitivity to small values of $\ell(p)$. With this in mind, we let

$$f(n) = \sum_{p \mid 2^n - 1} \frac{1}{p}.$$

Then the average order problem becomes tractable:

$$\begin{aligned}\sum_{n \le x} f(n) &= \sum_{n \le x} \sum_{p \mid 2^n - 1} \frac{1}{p} \\ &= \sum_{p > 2} \frac{1}{p} \sum_{\substack{n \le x \\ \ell(p) \mid n}} 1 \approx x \sum_{p > 2} \frac{1}{p\ell(p)}.\end{aligned}$$

It is not hard to prove that the sum converges and that the approximation is justified: $\frac{1}{x} \sum_{n \le x} f(n) \to \sum_{p > 2} \frac{1}{p\ell(p)}$.

The function $f(n)$ was introduced by Erdős, who was interested in large values of $f(n)$.

One way of constructing large values of $f(n)$ is to make n divisible by all of the small numbers. Choose $z = \frac{1}{2}\log x$, and let n be the lcm of all positive integers $\leq z$. Then $n \leq x$ (for large x). Moreover,

$$f(n) = \sum_{p \mid 2^n - 1} \frac{1}{p} = \sum_{\ell(p) \mid n} \frac{1}{p}$$

$$\geq \sum_{2 < p \leq z} \frac{1}{p}.$$

By a theorem of Mertens, for a certain constant $C_0 = 0.261\ldots$,

$$\sum_{p \leq z} \frac{1}{p} = \log\log z + C_0 + o(1)$$

$$= \log\log\log x + C_0 + o(1).$$

So we know that for all large x, there are values of $n \le x$ with

$$f(n) \ge \log\log\log x - \frac{1}{2} + C_0 + o(1).$$

In 1971, Erdős proved the remarkable result that this inequality is sharp up to the constant addend: For some constant C, all large real numbers x, and all $n \le x$,

$$f(n) \le \log\log\log x + C.$$

Paul Erdős

Theorem (Engberg, 2014)

Assume the GRH and the Elliott–Halberstam Conjecture. There is a constant $C_1 \approx 0.522$ *such that the following holds: For all large* x*, there are values of* $n \leq x$ *for which*

$$f(n) \geq \log\log\log x - \frac{1}{2} + C_0 + C_1 + o(1).$$

This is sharp, in the sense that the reverse inequality holds for all $n \leq x$*, as* $x \to \infty$*.*

Here $C_1 = \int_1^\infty \rho(u)u^{-1}\,du$, where $\rho(u)$ is Dickman's rho-function (stay tuned).

In the interests of time, I will focus on the lower bound implicit in the theorem.

Once again, take n the lcm of the positive integers not exceeding $z := \frac{1}{2}\log x$, so that $n \le x$. Then

$$f(n) \ge \sum_{2<p\le z} \frac{1}{p} \qquad\qquad .$$

In the interests of time, I will focus on the lower bound implicit in the theorem.

Once again, take n the lcm of the positive integers not exceeding $z := \frac{1}{2}\log x$, so that $n \le x$. Then

$$f(n) \ge \sum_{2<p\le z} \frac{1}{p} + \sum_{\substack{p>z \\ \ell(p)|n}} \frac{1}{p}.$$

Since $\ell(p) \mid p-1$, we can bound the remaining contribution from below:

$$\sum_{\substack{p>z \\ \ell(p)|n}} \frac{1}{p} \ge \sum_{\substack{p>z \\ p-1|n}} \frac{1}{p}.$$

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If $p-1 \mid n$, then $P(p-1) \leq z$. We argue that

$$\sum_{\substack{p>z \\ p-1|n}} \frac{1}{p} = \sum_{\substack{p>z \\ P(p-1)\leq z}} \frac{1}{p} + o(1).$$

To understand this last sum, we need to understand the frequency with which shifted primes $p-1$ have only small prime factors.

Dickman: For each fixed $u \geq 0$, the limiting proportion of $n \leq x$ with $P(n) \leq x^{1/u}$ exists. We call this $\rho(u)$; that is,

$$\rho(u) = \lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : P(n) \leq x^{1/u}\}.$$

The function $\rho(u)$ is positive but decays rapidly as $u \to \infty$, roughly like u^{-u}.

Granville: Assume the Elliott–Halberstam Conjecture. For each fixed $u \geq 0$, the limiting proportion of $p-1 \leq x$ with $P(p-1) \leq x^{1/u}$ is also given by $\rho(u)$.

Using Granville's theorem, we prove (under EHC) that the function of z given by

$$\sum_{\substack{p>z \\ P(p-1)\leq z}} \frac{1}{p}$$

converges as $z \to \infty$ to

$$\int_1^\infty \rho(u)u^{-1}\,du =: C_1.$$

Collecting estimates shows (under EHC) that for all large x, there is an integer $n \leq x$ with

$$f(n) \geq \log\log\log x - \frac{1}{2} + C_0 + C_1 + o(1).$$

In fact, we can take n as the lcm of the numbers $\leq \frac{1}{2}\log x$.

We prove that this is sharp by establishing that the same expression serves as an upper bound, valid for *all* $n \leq x$. How? Overarching arguments are similar, but now need GRH.

Why? We replaced $\ell(p)$ with $p-1$ above. GRH is used to show that this doesn't make much difference, since the ratio $(p-1)/\ell(p)$ is usually small.

The method also allows us to handle certain relatives of $f(n)$. For example, let

$$g(n) = \sum_{d|2^n-1} \frac{1}{d}.$$

Note that this is equal to

$$\sigma(2^n-1)/(2^n-1),$$

where σ is the usual sum-of-divisors function.

Assuming GRH and EHC, Zeb and I prove that as $x \to \infty$,

$$\max_{n \le x} g(n) \sim \frac{1}{2} e^{\gamma + C_1} \log\log x.$$

Thank you your attention!