Multiperfect numbers with identical digits

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Abstract

Let $g \geq 2$. A natural number N is called a *repdigit in base g* if all of the digits in its base g expansion are equal, i.e., if $N = D \cdot \frac{g^m - 1}{g-1}$ for some $m \geq 1$ and some $D \in \{1, 2, \ldots, g-1\}$. We call N *perfect* if $\sigma(N) = 2N$, where σ denotes the usual sum-of-divisors function. More generally, we call N *multiperfect* if $\sigma(N)$ is a proper multiple of N. The second author recently showed that for each fixed $g \geq 2$, there are finitely many repdigit perfect numbers in base g, and that when $g = 10$, the only example is $N = 6$. We prove the same results for repdigit multiperfect numbers.

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1. Introduction

Let $g \geq 2$. A natural number N is called a *repdigit in base* g if all of the digits in its base g expansion are equal; equivalently, N is a repdigit in base g if

$$N = D\frac{g^m - 1}{g - 1} \quad \text{for some} \quad m \geq 1, \quad D \in \{1, 2, \dots, g - 1\}. \tag{1}$$

Several authors have investigated the arithmetic properties of repdigits. For example, Bugeaud and Mignotte [1] have shown that the only repdigit perfect powers in base 10 are $1, 4, 8$, and 9. This settled an old problem of Obláth [2].

In this paper, we are concerned not with perfect powers but with perfect numbers. Let σ denote the familiar sum-of-divisors function. Recall that a natural number N is called *perfect* if $\sigma(N) = 2N$ and *multiperfect* (or *multiply perfect*) whenever $\sigma(N)$ is a proper multiple of N. In the latter case, the ratio $\ell = \sigma(N)/N$ is referred to as the *abundancy* of N, and N is called ℓ-perfect. The first example of a perfect number is $N = 6$, while the first example of an ℓ-perfect number with $\ell > 2$ is $N = 120$ (with $\ell = 3$). There are numerous unsolved problems concerning perfect and multiperfect numbers; see [3, Chapter 1] for an up-to-date survey, and the website [4] for a database of all known examples.

The second author [5] recently investigated repdigit perfect numbers. He showed that in each base g, there are only finitely many examples, and that when $g = 10$, the only example is $N = 6$. The method was that of the first author [6], who had earlier shown that there are no perfect Fibonacci numbers (see also [7]). Quite recently it was shown [8] that there are no multiply perfect Fibonacci numbers. Inspired by this achievement, we establish the following results:

Theorem 1. *Fix $g \geq 2$. There are only finitely many repdigit multiperfect numbers N in base g. Moreover, there is a computable upper bound on the number of such N.*

The conclusion of Theorem 1 is slightly weaker than that of the corresponding result for perfect repdigits in [5], where all examples were shown to bounded by an effective constant depending on g. Our argument here just barely fails to establish this; we obtain an effective bound whenever the number of digits is not a power of 2 (cf. the remarks on p. 122 of [7]).

In the case of most interest, when $g = 10$, we are able to reduce the computation to something manageable and so determine all multiperfect repdigits.

Theorem 2. *When $g = 10$, the only multiperfect repdigit is $N = 6$.*

Notation and conventions

We assume that g is an integer with $g \geq 2$. We write U_m and V_m for the Lucas sequences of the first and second kind, respectively, with roots g and 1. Thus, for each $m \geq 0$,

$$U_m := \frac{g^m - 1}{g - 1} \quad \text{and} \quad V_m := g^m + 1.$$

For each positive integer d coprime to g, we let $z(d)$ denote the *rank of appearance* of d for the U_m. In other words, $z(d)$ is the minimal natural number for which $d \mid U_{z(d)}$. (Thus, if p is a prime not dividing g, then $z(p)$ is the order of p modulo g, except when p divides $g-1$, when $z(p) = p$.) We take for granted basic properties of Lucas sequences, as described, e.g., in [9, Chapter 4].

Throughout, the letters p, P, q, Q, and r, with or without subscripts, denote primes. We write $\omega(m) := \sum_{p \mid m} 1$ for the number of distinct prime factors of the positive integer m, and we write $\Omega(m) := \sum_{p^\ell \mid m} 1$ for the number of prime factors of m counted with multiplicity. For a prime p and a positive integer m, we use $\nu_p(m)$ for the exponent of p in the factorization of m.

We use the Bachmann-Landau notation "$f = O(g)$" and the Vinogradov notation "$f \ll g$" interchangeably; both mean that there is an *effective* constant C with $|f| \leq Cg$. Subscripts indicate parameters on which the implied constants may depend.

We write e for the base of the natural logarithm.

2. General g: Proof of Theorem 1

Throughout this section, we assume that $g \geq 2$ is fixed. We are to show that there are only finitely many multiperfect numbers of the form $N = D\frac{g^m-1}{g-1}$, where $D \in \{1, 2, \ldots, g-1\}$.

2.1. The case $m = 2^s$.

We first treat the case when $m = 2^s$ for some s. Fix $D \in \{1, 2, \ldots, g-1\}$. Observe that $U_1 \mid U_2 \mid U_4 \mid U_8 \mid \ldots$. It follows that the sequence

$$\frac{\sigma(D \cdot U_{2^s})}{D \cdot U_{2^s}}, \quad \text{with} \quad s = 1, 2, 3, \ldots \tag{2}$$

is strictly increasing. Indeed, whenever a is a proper divisor of b, we have $\sigma(a)/a < \sigma(b)/b$; this is clear upon recalling that $\sigma(n)/n = \sum_{d \mid n} d^{-1}$ for each natural number n. Consequently, for each ℓ, there is at most one s for which $D \cdot U_{2^s}$ is multiperfect with abundancy ℓ. So it suffices to show that there is an effective bound on the possible values of ℓ. Since

$$\frac{\sigma(D \cdot U_{2^s})}{D \cdot U_{2^s}} \leq \frac{\sigma(D)}{D} \frac{\sigma(U_{2^s})}{U_{2^s}} \ll_g \frac{\sigma(U_{2^s})}{U_{2^s}},$$

we will have succeeded if we show that the ratios $\sigma(U_{2^s})/U_{2^s}$ are bounded. Actually we prove more: In general, $\sigma(U_m)/U_m$ is bounded in terms of g and the number of distinct prime factors of m. The proof requires a simple identity.

If $\mathcal{P}$ is a finite set of primes, we write $\mathcal{P}^*$ for the set of natural numbers all of whose prime factors belong to $\mathcal{P}$.

Lemma 1. *Let $\mathcal{P}$ be a finite set of primes. We have*

$$\sum_{n\in\mathcal{P}^*}\frac{\log n}{n}=\left(\sum_{p\in\mathcal{P}}\frac{\log p}{p-1}\right)\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{-1}.$$

Proof. We start by inserting the identity $\log n=\sum_{d\mid n}\Lambda(d)$, where Λ is the von Mangoldt function. Upon reversing the order of summation, we find that

$$\begin{aligned}\sum_{n\in\mathcal{P}^*}\frac{\log n}{n}&=\sum_{d\in\mathcal{P}^*}\Lambda(d)\sum_{\substack{n\in\mathcal{P}^*\\ d\mid n}}\frac{1}{n}\\ &=\sum_{d\in\mathcal{P}^*}\frac{\Lambda(d)}{d}\sum_{n'\in\mathcal{P}^*}\frac{1}{n'}.\end{aligned}$$

Now

$$\sum_{n'\in\mathcal{P}^*}\frac{1}{n'}=\prod_{p\in\mathcal{P}}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right)=\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{-1},$$

and

$$\sum_{d\in\mathcal{P}^*}\frac{\Lambda(d)}{d}=\sum_{\substack{p\in\mathcal{P}\\ k\geq 1}}\frac{\log p}{p^k}=\sum_{p\in\mathcal{P}}\frac{\log p}{p-1}.$$ ◻

We can now prove the bound alluded to above.

Lemma 2. *Let m be an integer with $m>1$. Then*

$$\log\frac{\sigma(U_m)}{U_m}\ll_g(\log(\mathrm{e}\omega(m)))^2.$$

Proof. Observe that

$$\begin{aligned}\frac{\sigma(U_m)}{U_m}&=\prod_{p^e\| U_m}\left(1+\frac{1}{p}+\dots+\frac{1}{p^e}\right)\\ &\leq\prod_{p\mid U_m}\left(1+\frac{1}{p-1}\right)\leq\exp\left(\sum_{p\mid U_m}\frac{1}{p-1}\right).\end{aligned}\tag{3}$$

For each prime p dividing U_m, the number $z(p)$ is a divisor of m with $z(p)>1$. Moreover, if $p\nmid g-1$, then $z(p)$ divides $p-1$. Hence,

$$\sum_{p\mid U_m}\frac{1}{p-1}\leq\sum_{p\mid g-1}\frac{1}{p-1}+\sum_{\substack{d\mid m\\ d>1}}\sum_{\substack{p\mid U_d\\ p\equiv 1\pmod d}}\frac{1}{p-1}.\tag{4}$$

Let d be a divisor of m with $d > 1$. Since $U_d < g^d$, the number of prime divisors of U_m from the progression 1 mod d is bounded by $\log(g^d)/\log d$, and so

$$\sum_{\substack{p \mid U_d \\ p \equiv 1 \pmod{d}}} \frac{1}{p-1} \le \sum_{1 \le k \le d \log g/\log d} \frac{1}{dk} \le \frac{1}{d}\left(1 + \log\left(\frac{d \log g}{\log d}\right)\right) \ll_g \frac{\log(\mathrm{e}d)}{d}. \tag{5}$$

Consequently,

$$\sum_{\substack{d \mid m \\ d > 1}} \sum_{\substack{p \mid U_d \\ p \equiv 1 \pmod{d}}} \frac{1}{p-1} \ll_g \sum_{d \mid m} \frac{\log(\mathrm{e}d)}{d}. \tag{6}$$

Write the prime factorization of m in the form $q_1^{e_1} \cdots q_k^{e_k}$, where $q_1 < q_2 < \cdots < q_k$. (Thus, $k = \omega(m)$.) If $d \mid m$, then we can write $d = q_1^{f_1} \cdots q_k^{f_k}$, with $0 \le f_i \le e_i$ for all $1 \le i \le k$. Let p_i denote the ith prime. Since $x \mapsto \log(\mathrm{e}x)/x$ is decreasing for $x \ge 1$, we have $\log(\mathrm{e}d)/d \le \log(\mathrm{e}d')/d'$, where $d' := p_1^{f_1} \cdots p_k^{f_k}$. It follows that with $\mathcal{P} := \{p_1, \dots, p_k\}$, the right-hand side of (6) is majorized by

$$\sum_{d' \in \mathcal{P}^*} \frac{\log(\mathrm{e}d')}{d'} = \left(1 + \sum_{p \in \mathcal{P}} \frac{\log p}{p-1}\right) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right)^{-1} \ll (\log(\mathrm{e}\omega(m)))^2, \tag{7}$$

using the prime number theorem and Mertens's theorems (explicit versions of which are available in, e.g., [10]). The lemma follows upon collecting estimates (3)–(7). □

2.2. The general case

Suppose now that m is not a power of 2. In this case, we are able to show that all examples are effectively bounded (and not merely the number of examples). Our argument rests on the following estimate for the 2-part of $\sigma(U_m)$:

Lemma 3. *Let m be a natural number, and write $m = 2^s n$, where n is odd. Suppose that $n > 1$. Then $\nu_p(U_m)$ is odd for at least $\Omega(n) + 2s + O_g(1)$ odd primes p.*

We defer the proof of Lemma 3 to the next section. Let us see how we may use it to deduce Theorem 1.

Suppose that $N = D\frac{g^m-1}{g-1}$ is multiperfect, and let ℓ be the abundancy of N. By Lemma 2,

$$\log \ell \le \log \frac{\sigma(D)}{D} + \log \frac{\sigma(U_m)}{U_m} \ll_g (\log(e\omega(m)))^2. \tag{8}$$

We wish to compare this with the trivial bound

$$\log \ell \ge \nu_2(\ell) \log 2. \tag{9}$$

Note that $\nu_2(\ell) = \nu_2(\sigma(N)) - \nu_2(N)$. Since D is divisible by only $O_g(1)$ primes, Lemma 3 implies that $\nu_p(D \cdot U_m)$ is odd for at least $\Omega(n) + 2s + O_g(1)$ odd primes p. Hence,

$$\nu_2(\sigma(N)) = \nu_2(\sigma(D \cdot U_m)) \geq \Omega(n) + 2s + O_g(1).$$

On the other hand, $\nu_2(N) = \nu_2(D) + \nu_2(U_m) = O_g(1) + \nu_2(U_m)$. Write

$$U_m = \frac{g^{2^s n} - 1}{g^{2^s} - 1}(g^{2^{s-1}} + 1) \cdots (g + 1). \tag{10}$$

Since n is odd,

$$\frac{g^{2^s n} - 1}{g^{2^s} - 1} = (g^{2^s})^{n-1} + (g^{2^s})^{n-2} + \cdots + 1 \equiv (n-1)g^{2^s} + 1 \equiv 1 \pmod 2,$$

and so the first right-hand factor in (10) makes no contribution to $\nu_2(U_m)$. If g is even, then the remaining s factors in (10) are also odd; otherwise, for all $1 \leq i \leq s - 1$,

$$g^{2^i} + 1 \equiv 2 \pmod 4, \quad \text{so that} \quad \nu_2(g^{2^i} + 1) = 1.$$

So regardless of the parity of g,

$$\nu_2(U_m) \leq s + O_g(1).$$

Thus,

$$\begin{aligned} \nu_2(\ell) &= \nu_2(\sigma(N)) - \nu_2(N) \\ &\geq \Omega(n) + s + O_g(1) = \Omega(m) + O_g(1). \end{aligned}$$

Combining this with (8) and (9) shows that

$$\omega(m) \leq \Omega(m) \ll_g (\log(\mathrm{e}\omega(m)))^2. \tag{11}$$

This estimate is key: It implies that $\omega(m)$ and $\Omega(m)$ are both bounded, and (8) now implies that ℓ is also bounded. The next lemma will allow us to bound the prime factors of m.

Lemma 4. *Let $G \geq 2$. Let $\epsilon > 0$, and let Z be a positive integer. Suppose that k is a natural number with $\omega(k) \leq Z$ for which*

$$\frac{\sigma((G^k - 1)/(G - 1))}{(G^k - 1)/(G - 1)} > 1 + \epsilon. \tag{12}$$

Then the smallest prime dividing k is bounded by a computable number depending on G, ϵ, and Z.

Proof. Assume that every prime factor of k exceeds W, where W is a large natural number to be specified in due course. We will show that we obtain a contradiction for large enough W. For each prime p, let $z'(p)$ denote the rank of appearance of p for $U'_m := \frac{G^m-1}{G-1}$. Then (cf. the proof of Lemma 2)

$$\log\left(\frac{\sigma((G^k-1)/(G-1))}{(G^k-1)/(G-1)}\right) \le \sum_{\substack{d\mid k\\ d>1}} \sum_{p:\ z'(p)=d} \frac{1}{p-1}.$$

Suppose that $W > G$. Then $z'(p)$ divides $p-1$ for each prime p dividing k. (Otherwise, $z'(p) = p$ is a prime divisor of both k and $G-1$, contradicting that $p > W > G$.) Following the proof of Lemma 2, we deduce that

$$\sum_{\substack{d\mid k\\ d>1}} \sum_{p:\ z'(p)=d} \frac{1}{p-1} \ll_G \sum_{\substack{d\mid k\\ d>1}} \frac{\log(\mathrm{e}d)}{d}.$$

Let $\mathcal{P}$ be the set of the first Z consecutive primes exceeding W. By Lemma 1,

$$\begin{aligned}\sum_{\substack{d\mid k\\ d>1}} \frac{\log(\mathrm{e}d)}{d} &\le \sum_{\substack{d\in\mathcal{P}^*\\ d>1}} \frac{\log(\mathrm{e}d)}{d}\\ &= \left(\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{-1} - 1\right) + \left(\sum_{p\in\mathcal{P}} \frac{\log p}{p-1}\right)\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{-1}.\end{aligned}$$

If we add to our conditions on W that $W \ge 2Z$, we find that

$$1 \le \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{-1} \le \exp\left(\sum_{p\in\mathcal{P}}\frac{1}{p-1}\right) \le \exp\left(\frac{Z}{W}\right) \le 1 + O\left(\frac{Z}{W}\right).$$

Also, since $\log x/(x-1)$ is decreasing for $x > 1$,

$$\sum_{p\in\mathcal{P}} \frac{\log p}{p-1} \le Z\frac{\log(W+1)}{W}.$$

Putting everything together, we find that

$$\log\left(\frac{\sigma((G^k-1)/(G-1))}{(G^k-1)/(G-1)}\right) \ll_G Z\frac{\log(W+1)}{W}.$$

Now choosing W sufficiently large, we obtain a contradiction with (12). $\square$

We have seen already that $\Omega(m)$ is effectively bounded in terms of g, see (11). Fix such a bound Z, so that

$$\Omega(m) \le Z.$$

We can now complete the proof of Theorem 1 (still assuming Lemma 3). Since ℓ is bounded, we may fix ℓ and restrict attention to ℓ-perfect numbers. Since there are only $O_g(1)$ choices for the digit D, we also assume that D is fixed. We proceed inductively: We show that for each $j = 0, 1, 2, 3, \dots,$ the following assertion holds:

$$\boxed{\text{Let } N = D \cdot \tfrac{g^m-1}{g-1} \text{ be an } \ell\text{-perfect number, where } m \text{ is not a power of 2. Suppose that } \Omega(N) \geq j. \text{ Then the } j \text{ smallest (not necessarily distinct) prime factors of } m \text{ are bounded. The bound here depends at most on } D, g, j, \text{ and } \ell.} \qquad (A_j)$$

Since $\Omega(m)$ is bounded by Z, this proves Theorem 1.

Statement A_0 is vacuously true. Assume that A_j is known to be true, and take an ℓ-perfect number N of the prescribed form with $\Omega(N) \geq j+1$. Write $m = p_1 p_2 \cdots p_j m'$, where $p_1 \leq p_2 \leq \cdots \leq p_j$ is the list of the j smallest prime factors of m. Then

$$\ell = \frac{\sigma(N)}{N} = \frac{\sigma(D \cdot U_{m/m'} \frac{U_m}{U_{m/m'}})}{D \cdot U_{m/m'} \frac{U_m}{U_{m/m'}}} \leq \frac{\sigma(D \cdot U_{m/m'})}{D \cdot U_{m/m'}} \frac{\sigma(U_m/U_{m/m'})}{U_m/U_{m/m'}}. \tag{13}$$

Since $m' > 1$, the number $D \cdot U_{m/m'}$ is a proper divisor of N, and so

$$\frac{\sigma(D \cdot U_{m/m'})}{D \cdot U_{m/m'}} < \frac{\sigma(N)}{N} = \ell. \tag{14}$$

In fact, we can say something a bit stronger. By the induction hypothesis, there are only finitely many possibilities for m/m'. Consequently, the left hand-side of (14) is bounded away from ℓ; in other words, for some small $\epsilon > 0$ (depending only on $D, g, j,$ and ℓ), we have

$$\frac{\sigma(D \cdot U_{m/m'})}{D \cdot U_{m/m'}} < \ell(1-\epsilon),$$

uniformly in the original choice of N. By (13),

$$\frac{\sigma(U_m/U_{m/m'})}{U_m/U_{m/m'}} \geq \ell \frac{D \cdot U_{m/m'}}{\sigma(D \cdot U_{m/m'})} > \frac{1}{1-\epsilon} > 1 + \epsilon. \tag{15}$$

Set $G := g^{m/m'}$. Then

$$U_m/U_{m/m'} = \frac{g^m - 1}{g^{m/m'} - 1} = \frac{G^{m'} - 1}{G - 1}. \tag{16}$$

Clearly $\omega(m') \leq Z$. By (15), (16), and Lemma 12, the least prime factor of m' is bounded by a constant, depending on $g^{m/m'}$, ϵ, and Z. Here Z depends only on g. Also, ϵ depends only on $D, g, j,$ and ℓ, and there are only finitely many possibilities for m/m'. Thus, the least prime factor of m', which is the $(j+1)$th smallest prime factor of m, is bounded by a quantity depending only on D, g, $j+1$, and ℓ. This completes the induction.

2.3. Proof of Lemma 3

We need some preliminary lemmas. Let us write $\Box$ to denote a rational perfect square, i.e., a generic element of $(\mathbf{Q}^{\times})^2$.

Lemma 5 (Ljunggren [11])**.** *The only integer solutions (G,k) with $|G|>1$ and $k>2$ to the exponential Diophantine equation*

$$\frac{G^k-1}{G-1}=\Box$$

are $(G,k)=(7,4)$ and $(G,k)=(3,5)$.

Recall that for $\mathcal{P}$ a finite set of primes, $\mathcal{P}^*$ denotes the set of natural numbers all of whose prime factors belong to $\mathcal{P}$.

Lemma 6. *Let $G\geq 2$. Let $\mathcal{P}$ be a finite set of prime numbers. The set of k for which $G^k+1=A\Box$ for some $A\in\mathcal{P}^*$ is a finite set. Moreover, all such k are bounded by an effective constant depending only on G and $\mathcal{P}$.*

The next lemma is implicit in the proof of [5, Lemma 8].

Lemma 7. *Let $G\geq 2$. Let $\mathcal{P}$ be a finite set of prime numbers, and let $r>1$. The set of k for which*

$$\frac{G^{kr}-1}{G^k-1}=A\Box \tag{17}$$

for some $A\in\mathcal{P}^*$ is a finite set. Moreover, all such k are bounded by an effective constant depending only on $G,\mathcal{P}$, and r.

Proof. Putting $x=G^k$, we can rewrite (17) in the form

$$x^{r-1}+x^{r-2}+\cdots+1=A\Box.$$

The left-hand expression is a polynomial in x with $r-1$ simple roots. If $r-1\geq 3$, then the desired result follows immediately from an effective version of Siegel's theorem (see, e.g., [12, Theorem 6.2]). So we need only consider the cases $r=2$ and $r=3$. If $r=2$, then (17) takes the form $G^k+1=A\Box$, and the result follows from Lemma 6. So suppose that $r=3$. Write $G^k=G^{\delta}y^2$, where $\delta\in\{0,1\}$ and y is an integer. Then (17) implies that y is a solution to

$$G^{2\delta}y^4+G^{\delta}y^2+1=A\Box.$$

It is simple to check that for both $\delta=0$ and $\delta=1$, the left-hand polynomial in y has four simple roots, so that another appeal to Siegel's theorem finishes the proof. $\Box$

Proof of Lemma 3. Let $\omega'(\cdot)$ stand for the additive function which counts the number of odd primes which appear to an odd exponent. We are to show that, if $m=2^s n$ with $n>1$ and odd, then

$$\omega'(U_m)\geq\Omega(n)+2s+O_g(1).$$

By repeated application of the identity $U_{2i} = U_i V_i$, we obtain a fatorization $U_m = UV$, where

$$U := U_n \quad \text{and} \quad V := V_n V_{2n} \cdots V_{2^{s-1}n}.$$

Here U and V are relatively prime: Indeed, if p divides U, then $g^n \equiv 1 \pmod{p}$; this implies that for each $0 \le i < s$,

$$V_{2^i n} = g^{2^i n} + 1 \equiv 2 \pmod{p},$$

and so $p \nmid V$ except possibly if $p = 2$. But U is odd since n is odd, so that $\gcd(U, V) = 1$, as claimed. It follows that to bound $\omega'(U_m)$, is it enough to bound $\omega'(U)$ and $\omega'(V)$.

We start with U. Write the prime factorization of n in the form $n = p_1 p_2 \cdots p_k$, where $p_1 \le p_2 \le \cdots \le p_k$. Put $n_1 := n$, and for $1 \le i \le k$, successively define $n_{i+1} := n_i/p_i$. (Thus $n_i = p_i \cdots p_k$, with $n_{k+1} = 1$.) We have the k-fold decomposition

$$U = \frac{U_{n_1}}{U_{n_2}} \frac{U_{n_2}}{U_{n_3}} \cdots \frac{U_{n_k}}{U_{n_{k+1}}}. \tag{18}$$

Suppose that p is a prime which divides two of the right-hand factors, say the ith and jth, where $i < j$. Then $n_j \mid n_{i+1}$, so that

$$p \mid \frac{U_{n_j}}{U_{n_{j+1}}} \mid g^{n_j} - 1 \mid g^{n_{i+1}} - 1.$$

Thus $g^{n_{i+1}} \equiv 1 \pmod{p}$. But also

$$p \mid \frac{U_{n_i}}{U_{n_{i+1}}} = \frac{g^{n_i} - 1}{g^{n_{i+1}} - 1},$$

so that modulo p,

$$\begin{aligned} 0 \equiv \frac{g^{n_i} - 1}{g^{n_{i+1}} - 1} = \frac{g^{n_{i+1} p_i} - 1}{g^{n_{i+1}} - 1} &= 1 + g^{n_{i+1}} + \cdots + g^{(p_i - 1) n_{i+1}} \\ &\equiv 1 + 1 + \cdots + 1 \equiv p_i. \end{aligned}$$

Hence, $p = p_i$. Since $p \mid U_{n_j}/U_{n_{j+1}} \mid U_{n_j}$, we have that $z(p)$ divides $n_j = p_j \cdots p_k$. But $z(p) = z(p_i)$ is divisible only by primes $\le p_i$, while n_j is divisible only by primes $\ge p_j$. Since $i < j$, this is only possible if $z(p_i) = p_i$, so that $p = p_i$ divides $g - 1$. Moreover, $p_i = p_{i+1} = \cdots = p_j$.

This suggests a division of the indices $1 \le i \le k$ into classes. For each prime p dividing $\gcd(n, g-1)$, let $\mathcal{C}_p$ denote the set of indices i for which $p_i = p$, and let $\mathcal{C}_0$ consist of the indices not belonging to any $\mathcal{C}_p$. Then if i and j belong to distinct classes, $\gcd(U_{n_i}/U_{n_{i+1}}, U_{n_j}/U_{n_{j+1}}) = 1$, and the same holds for distinct indices i and j both belonging to $\mathcal{C}_0$. We claim that for each class $\mathcal{C}$,

$$\omega'\left(\prod_{i \in \mathcal{C}} U_{n_i}/U_{n_{i+1}}\right) \ge \#\mathcal{C} + O_g(1). \tag{19}$$

Since there are only $O_g(1)$ classes, and the union of the classes has size k, this shows that

$$\omega'(U) \geq k + O_g(1) = \Omega(n) + O_g(1). \tag{20}$$

Suppose that $\mathcal{C} = \mathcal{C}_0$. Lemma 5 shows that there is at most one $i \in \mathcal{C}$ for which

$$U_{n_i}/U_{n_{i+1}} = \frac{g^{n_{i+1}p_i} - 1}{g^{n_{i+1}} - 1}$$

is a square. (Note that such an i gives rise to a solution of Ljunggren's equation with $G = g^{n_{i+1}}$ and $k = p_i$.) Since $U_{n_i}/U_{n_{i+1}}$ is odd, we find that

$$\omega'(U_{n_i}/U_{n_{i+1}}) \geq 1$$

for all $i \in \mathcal{C}$, with at most one exception. Since $U_{n_i}/U_{n_{i+1}}$ and $U_{n_j}/U_{n_{j+1}}$ are coprime for distinct $i, j \in \mathcal{C}_0$, this proves the claim (19) when $\mathcal{C} = \mathcal{C}_0$.

Now suppose that $\mathcal{C} = \mathcal{C}_p$, where $p \mid \gcd(n, g-1)$. For every two indices $i, j \in \mathcal{C}$, the greatest common divisor of $U_{n_i}/U_{n_{i+1}}$ and $U_{n_j}/U_{n_{j+1}}$ is supported on the primes dividing $g - 1$. So to prove (19), it is enough to show that for all but $O_g(1)$ indices $i \in \mathcal{C}$, there is a prime p not dividing $g - 1$ for which $\nu_p(U_{n_i}/U_{n_{i+1}})$ is odd. Since

$$U_{n_i}/U_{n_{i+1}} = \frac{g^{n_{i+1}p} - 1}{g^{n_{i+1}} - 1},$$

this follows from Lemma 7, taking $G = g$, $r = p$, and $\mathcal{P}$ the set of primes dividing $g - 1$. This proves the claim for these $\mathcal{C}$ and completes the proofs of (19) and (20).

We now turn to estimating $\omega'(V)$. We will show that

$$\omega'(V) \geq 2s + O_g(1),$$

which with (20) completes the proof of the lemma. (Recall that U and V are relatively prime.)

In the decomposition $V = V_n V_{2n} \cdots V_{2^{s-1}n}$, no pair of the s factors has a common odd prime divisor. Indeed, if p is an odd prime dividing $V_{2^i n}$ for some $0 \leq i < s$, then $g^{2^i n} \equiv -1 \pmod{p}$, which implies that g^n has order 2^{i+1} modulo p. Consequently, the index i is uniquely determined. So it suffices to show that for all but $O_g(1)$ indices i, with $0 \leq i < s$, we have $\omega'(V_{2^i n}) \geq 2$.

Fix q as the smallest prime divisor of n, and write

$$V_{2^i n} = \frac{V_{2^i n}}{V_{2^i n/q}} V_{2^i n/q}.$$

(Here we use our hypothesis that $n > 1$.) If we rewrite the first term on the right-hand side in the form

$$\frac{V_{2^i n}}{V_{2^i n/q}} = \frac{(-g^{2^i n/q})^q - 1}{-g^{2^i n/q} - 1}, \tag{21}$$

then we see from Lemma 5 that $V_{2^i n}/V_{2^i n/q}$ is never a square. Since q is odd, $V_{2^i n}/V_{2^i n/q}$ is odd, and so

$$\omega'(V_{2^i n}/V_{2^i n/q}) \geq 1. \tag{22}$$

Also, by Lemma 6 (with $\mathcal{P} = \{2\}$ and $G = g$),

$$\omega'(V_{2^i n/q}) \geq 1 \tag{23}$$

for all but $O_g(1)$ indices i. Finally, we claim that there is at most one index i for which $V_{2^i n}$ and $V_{2^i n/q}$ are not relatively prime. To see this, suppose that r is a common prime factor. Since $r \mid V_{2^i n/q}$, we have $g^{2^i n/q} \equiv -1 \pmod{r}$; thus

$$0 \equiv \frac{V_{2^i n}}{V_{2^i n/q}} \equiv 1 + 1 + \cdots + 1 \equiv q \pmod{r},$$

so that $r = q$. But we have seen already that q divides $V_{2^i n}$ for at most one value of i, and so the claim is proved. Now (22) and (23) yield that $\omega'(V_{2^i n}) \geq 2$ for all but $O_g(1)$ indices i with $0 \leq i < s$, as desired. □

3. A fundamental inequality

For the remainder of the paper, we fix $g = 10$. For each integer $m > 0$, let $\omega''(m)$ denote the number of primes $p \equiv 3 \pmod{4}$ for which $\nu_p(m)$ is odd. The following lemma is fundamental for the proof of Theorem 2.

Lemma 8. *Let $m > 1$ be an integer, and put $k = \omega(m)$. Then*

$$\omega''(U_m) \geq 2^k - 2k. \tag{24}$$

If m is coprime to 3, then the above inequality is strict.

3.1. Proof of Lemma 8

We first claim that if $d > 1$ is squarefree, then

$$\Phi_d(X) \equiv 1 \pm X \pmod{X^2}, \tag{25}$$

where $\Phi_d(X)$ denotes the dth cyclotomic polynomial. By definition, $\Phi_d(X) = \prod_\zeta (X - \zeta)$, where ζ ranges over the primitive dth roots of unity. That $\Phi_d(X)$ has constant term 1 is immediate for $d = 2$, while for $d > 2$, this follows upon pairing ζ with ζ^{-1}. Moreover, the X-coefficient of $\Phi_d(X)$ is given by

$$-\sum_{\substack{1 \leq j \leq d \\ \gcd(j,d)=1}} \mathrm{e}^{2\pi i j/d} = -\mu(d).$$

(See, e.g., [13, §16.6].) This proves (25).

Now write

$$U_m = \prod_{d\in\mathcal{A}} \Phi_d(10) \prod_{d\in\mathcal{B}} \Phi_d(10), \tag{26}$$

where $\mathcal{A}$ consists of the squarefree divisors > 1 of m, and $\mathcal{B}$ consists of the remaining divisors > 1 of m. We infer from (25) that $\Phi_d(10) \equiv 3 \pmod 4$ for all $d \in \mathcal{A}$. Defining

$$\mathcal{P}_d := \{P \equiv 3 \pmod 4,\ \nu_P(\Phi_d(10)) \equiv 1 \pmod 2\},$$

it follows that $\#\mathcal{P}_d \geq 1$ for each $d \in \mathcal{A}$.

Recall that if a and b are positive integers with $a < b$, then

$$\gcd(\Phi_a, \Phi_b) = 1 \text{ or } P, \tag{27}$$

where P is prime; moreover, the second case occurs precisely when $a = P^\alpha z(P)$ and $b = P^\beta z(P)$ for some exponents $\beta > \alpha \geq 0$ (see the remarks preceding [14, Lemma 6]). So if a and b are both squarefree and $\Phi_a(10)$ and $\Phi_b(10)$ are not coprime, then $a = z(P)$ and $b = Pz(P)$ for some prime $P > 5$ for which $z(P)$ is squarefree.

Let $\mathcal{P}$ denote the union of the sets $\mathcal{P}_d$, for $d \in \mathcal{A}$. Clearly,

$$\begin{aligned}\#\mathcal{P} &\geq \sum_{d\in\mathcal{A}} \#\mathcal{P}_d - \sum_{\substack{d<e\\ d,\ e\in\mathcal{A}}} \#(\mathcal{P}_d \cap \mathcal{P}_e)\\ &\geq (2^k - 1) - \sum_{\substack{d<e\\ d,\ e\in\mathcal{A}}} \#(\mathcal{P}_d \cap \mathcal{P}_e).\end{aligned}$$

By the above, if $\mathcal{P}_d \cap \mathcal{P}_e \neq \emptyset$, then $\mathcal{P}_d \cap \mathcal{P}_e$ consists of only a single prime $P \geq 7$ and $(d, e) = (z(P),\ Pz(P))$. Thus, the prime P (which necessarily divides m) uniquely determines the pair (d, e). Observe also that P cannot be the smallest prime in m. Consequently,

$$\sum_{\substack{d<e\\ d,\ e\in\mathcal{A}}} \#(\mathcal{P}_d \cap \mathcal{P}_e) \leq k - 1,$$

so that $\#\mathcal{P} \geq 2^k - k$.

Let P be a prime in $\mathcal{P}$. If $P \nmid m$, then (by the above) P belongs to precisely one of the sets $\mathcal{P}_d$, with $d \in \mathcal{A}$, and so P shows up to an odd power in $\prod_{d\in\mathcal{A}} \Phi_d(10)$. Also, in this case, P does not divide $\prod_{d\in\mathcal{B}} \Phi_d(10)$. Hence, P appears to an odd power in U_m. The number of prime factors of m belonging to $\mathcal{P}$ is $\leq k$ always, and is $\leq k-1$ when m is coprime to 3, because when $3 \nmid m$, the smallest prime factor in m cannot belong to $\mathcal{P}$. Thus

$$\begin{aligned}\omega''(U_m) &\geq \#\mathcal{P} - k\\ &\geq 2^k - 2k,\end{aligned} \tag{28}$$

and the final inequality is strict if m is coprime to 3.

3.2. Perfect repdigits revisited

As a warm-up for the proof of Theorem 2, we offer another proof of the second principal result of [5].

Theorem A. *In base* 10*, the only perfect repdigit is* $N = 6$.

Proof. If N is perfect and even, then N has the form $2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are primes. Writing N in the form (1), we see that

$$p - 1 = \nu_2(N) = \nu_2(D) \leq 3,$$

so that $p \leq 4$. Thus, $p \in \{2, 3\}$ and $N \in \{6, 28\}$; but 28 is not a repdigit.

To show that there are no odd perfect numbers which are repdigits, we appeal to a classical result of Euler: Every odd perfect number N has the shape

$$N = p\square, \quad \text{where} \quad p \equiv 1 \pmod 4;$$

here p is referred to as the *special prime*. Suppose now that N is an odd perfect repdigit and write $N = D \cdot U_m$ with $D \in \{1, 2, \ldots, 9\}$. Clearly $m > 1$. Put $k = \omega(m)$. We take two cases.

Case I: $3 \nmid m$**.** Then D and U_m are coprime. Lemma 8 now shows that

$$\omega''(N) = \omega''(D \cdot U_m) \geq \omega''(U_m) \geq 2^k - 2k + 1 \geq 1.$$

But this contradicts the quoted result of Euler, according to which $\omega''(N) = 0$.

Case II: $3 \mid m$**.** In this case, U_m is a multiple of $U_3 = 3 \cdot 37$. Since D has at most one prime factor which is 3 modulo 4, we get that $\omega''(N) \geq 2^k - 2k - 1$. This expression is positive if $k \geq 3$, which forces $k = 1$ or $k = 2$.

Suppose that 37 is the special prime dividing N. Then $\nu_{37}(N)$ is odd, whence

$$19 \mid (37 + 1) \mid \sigma(N) = 2N.$$

Thus, $19 \mid U_m$, so that $18 = z(19) \mid m$. Since $k \leq 2$, we get that $m = 2^a 3^b$ for some positive integers a and b. Moreover, $U_{18} \mid N$. Since $11 \mid U_{18}$ and $N = 37\square$, it follows that $11^2 \mid N$. Hence, $11^2 \mid U_m$ and $2 \cdot 11 = z(11^2) \mid m$, which is a contradiction.

So assume that 37 is not the special prime. Then $37^2 \mid N$, so that $3 \cdot 37 = z(37^2) \mid m$. Since $k \leq 2$, this shows that $m = 3^a 37^b$ for some positive integers a and b. Since

$$2028119 \parallel U_{37} \mid N,$$

and 2028119 is a prime congruent to 3 modulo 4, we get that $2028119^2 \mid N$. Therefore $2028119 \mid z(2028119^2) \mid m$, which is again a contradiction. $\square$

4. $g = 10$: Proof of Theorem 2

Suppose that N is a multiply perfect repdigit in base $g = 10$. Write

$$N = D\frac{10^m - 1}{9}, \quad \text{where} \quad D \in \{1, 2, \ldots, 9\}.$$

We let ℓ denote the abundancy of N and we let k denote the number of distinct prime factors of m.

The case $\ell = 2$ is handled in Theorem A, so we assume that $\ell \geq 3$. A quick calculation with Mathematica shows that $m \geq 70$. Our plan is to iteratively bound k and ℓ until we get a contradiction.

We start by showing that

$$\nu_2(\ell) \geq 2^{k+1} - 4k - 3. \tag{29}$$

By Lemma 8, we have $\omega''(U_m) \geq 2^k - 2k$. So $\omega''(N) \geq 2^k - 2k - \omega(\gcd(D, U_m))$. Hence,

$$\begin{aligned} \nu_2(\sigma(N)) &\geq 2\omega''(N) \\ &\geq 2^{k+1} - 4k - 2\omega(\gcd(D, U_m)). \end{aligned}$$

Since $\nu_2(\sigma(N)) = \nu_2(\ell) + \nu_2(D)$, we get that

$$\nu_2(\ell) \geq 2^{k+1} - 4k - 2\omega(\gcd(D, U_m)) - \nu_2(D).$$

Thus, (29) follows once it is shown that

$$2\omega(\gcd(D, U_m)) + \nu_2(D) \leq 3. \tag{30}$$

If $D \in \{1, 2, 4, 5, 8\}$, then $\gcd(D, U_m) = 1$ and $\nu_2(D) \leq \nu_2(8) = 3$, yielding (30). If $D \in \{3, 7, 9\}$, then $\omega(D) = 1$ and $\nu_2(D) = 0$, and so again (30) holds. Finally, if $D = 6$, then $\omega(\gcd(D, U_m)) \leq \omega(3) = 1$ and $\nu_2(D) = 1$, and again we have (30).

4.1. Initial upper bounds on k and ℓ

From inequality (29),

$$\log \ell \geq (2^{k+1} - 4k - 3) \log 2. \tag{31}$$

On the other hand, proceeding as in the proof of Lemma 2, we see that

$$\begin{aligned} \log \ell = \log \frac{\sigma(N)}{N} &\leq \log \frac{\sigma(D)}{D} + \log \frac{\sigma(U_m)}{U_m} \\ &\leq \log \frac{15}{8} + \sum_{Q \mid U_m} \log\left(1 + \frac{1}{Q - 1}\right). \end{aligned} \tag{32}$$

To bound the remaining sum, we group the prime factors Q of U_m according to the value of $z(Q)$. Put $\mathcal{Q}_d = \{Q \;:\; z(Q) = d\}$. Since $z(Q) \mid Q - 1$ for all $Q \neq 3$, we have that $\mathcal{Q}_d$ consists of primes $p \equiv 1 \pmod d$ for all $d \neq 3$. Moreover, by a direct computation, $\mathcal{Q}_3 = \{3, 37\}$. In particular, putting $h := \#\mathcal{Q}_d$, the inequalities

$$h!d^h \leq \prod_{Q \in \mathcal{Q}_d} Q \leq \Phi_d(10) \leq 11^{\varphi(d)}$$

hold for all $d \geq 2$. Putting $\rho_0 = 1$ and $\rho_h = h!^{1/h}$ for $h \geq 1$, it follows that

$$h \leq \frac{\varphi(d) \log 11}{\log(\rho_h d)}.$$

Now it is straightforward to check that ρ_h is a nondecreasing function of h for nonnegative integers h. Thus, considering separately the cases when $h \leq 30$ and $h > 30$, we see that for $d \neq 3$,

$$\begin{aligned} S_d := \sum_{Q \in \mathcal{Q}_d} \frac{1}{Q-1} &\leq \frac{1}{d} \sum_{i=1}^{h} \frac{1}{i} \\ &\leq \frac{1}{d} \max \left\{ \sum_{i=1}^{30} \frac{1}{i}, \log \left(\mathrm{e} \frac{\varphi(d) \log 11}{\log(\rho_{31} d)} \right) \right\}. \end{aligned} \tag{33}$$

If $d \geq 55$, we have $\sum_{i=1}^{30} \frac{1}{i} < \log d$. For such d we also have

$$\mathrm{e} \frac{\varphi(d) \log 11}{\log(\rho_{31} d)} < \varphi(d) \frac{\mathrm{e} \log 11}{\log(55 \rho_{31})} < \varphi(d) < d.$$

Thus, for $d \geq 55$, we have $S_d < \frac{\log d}{d}$. Treating those $d < 55$ by direct calculation, we find that

$$S_d < \frac{\log d}{d} \qquad \text{for all} \qquad d \neq 3. \tag{34}$$

From (32), (34), and an appeal to the inequality $\log(1+x) < x$, we obtain that

$$\log \ell \leq \log \left(\frac{15}{8} \right) + \left\{ \log \left(1 + \frac{1}{2} \right) + \log \left(1 + \frac{1}{36} \right) - \frac{\log 3}{3} \right\} + \sum_{d \mid m} \frac{\log d}{d}.$$

We thus arrive at the numerically explicit bound

$$\log \ell < 0.7 + \sum_{d \mid m} \frac{\log d}{d}.$$

Let $\mathcal{P}$ be the set of primes dividing m. By Lemma 1,

$$\sum_{d \mid m} \frac{\log d}{d} \leq \sum_{d \in \mathcal{P}^*} \frac{\log d}{d} = \prod_{p \mid m} \left(1 + \frac{1}{p-1} \right) \left\{ \sum_{p \mid m} \frac{\log p}{p-1} \right\}.$$

Since $x \mapsto \frac{1}{x-1}$ and $x \mapsto \frac{\log x}{x-1}$ are both decreasing for $x > 1$, we get that (with p_k denoting the kth prime)

$$\log \ell < 0.7 + \prod_{p \le p_k} \left(1 + \frac{1}{p-1}\right) \left\{ \sum_{p \le p_k} \frac{\log p}{p-1} \right\}. \tag{35}$$

Comparing this with (31) shows that

$$(2^{k+1} - 4k - 3) \log 2 < 0.7 + \prod_{p \le p_k} \left(1 + \frac{1}{p-1}\right) \left\{ \sum_{p \le p_k} \frac{\log p}{p-1} \right\}. \tag{36}$$

This inequality fails for all $k \ge 5$. To see why, we use some explicit estimates from prime number theory: By inequalities (3.24) and (3.30) of [10, p. 70], we have for $k \ge 5$,

$$\prod_{p \le p_k} \left(1 + \frac{1}{p-1}\right) < 1.782 (\log p_k) \left(1 + \frac{1}{(\log p_k)^2}\right) < 1.782 \log p_k + 0.8,$$

and

$$\sum_{p \le p_k} \frac{\log p}{p-1} < \sum_{p \le p_k} \frac{\log p}{p} + \sum_{p} \frac{\log p}{p(p-1)} < \log p_k + 0.8.$$

Thus,

$$(2^{k+1} - 4k - 3) \log 2 < 0.7 + (1.782 \log p_k + 0.8)(\log p_k + 0.8). \tag{37}$$

Inserting the inequality $p_k < 2k \log k$ (see [10, inequality (3.13)]) into (37), we get an inequality which fails for all $k \ge 5$.

Thus $k \le 4$. Inequality (35) now shows that that $\log \ell < 9.32$, so that $\ell \le 11158$.

4.2. Reducing the bounds on k and ℓ

Let K and L denote known upper bounds on k and ℓ. At this point, we may take

$$K = 4, \qquad L = 11158.$$

Our goal in this section is to reduce these bounds. Given L, we let $V = V(L) := \lfloor \frac{\log L}{\log 2} \rfloor$, so that presently

$$V = 13.$$

Note that $\nu_2(\ell) \le V$ and $\nu_2(N) \le V + 3$.

Put $\mathcal{R} = \{3, 487, 56598313\}$. A calculation with Mathematica reveals that the primes in $\mathcal{R}$ are the only primes $r < 10^{10}$ for which $r^2 \mid 10^{r-1} - 1$. In particular, if $r < 10^{10}$ and $r^2 \mid U_m$ for some positive integer m, then either $r \in \mathcal{R}$ or $r \mid m$. Let $Z \ge 20$ be an integer parameter to be chosen later.

Write

$$N = N_1 N_2 N_3 N_4 N_5 N_6,$$

with

$$N_i = \prod_{\substack{p^{a_p} \| m \\ p \in \mathcal{A}_i}} p^{a_p} \qquad \text{for all} \qquad i = 1, \ldots, 6,$$

where

(i) $\mathcal{A}_1 = \mathcal{R}$,

(ii) $\mathcal{A}_2 = \{p \mid D\} \setminus \mathcal{A}_1$,

(iii) $\mathcal{A}_3 = \{p < 10^{10}\} \setminus (\mathcal{A}_1 \cup \mathcal{A}_2)$,

(iv) $\mathcal{A}_4 = \{10^{10} < p < 10^Z\}$,

(v) $\mathcal{A}_5 = \{p > 10^Z \ : \ p > z(p)^{5/2}\}$,

(vi) $\mathcal{A}_6 = \{p > 10^Z \ : \ p \le z(p)^{5/2}\}$.

Then

$$\log \ell = \log \frac{\sigma(N)}{N} = \sum_{i=1}^{6} \log \frac{\sigma(N_i)}{N_i}.$$

We estimate the contribution from each of the six right-hand terms separately.

To start off,

$$\log \frac{\sigma(N_1)}{N_1} < \sum_{r \in \mathcal{R}} \log \left(1 + \frac{1}{r-1}\right) < 0.408. \tag{38}$$

The set of primes in $\mathcal{A}_2$ dividing U_m is either empty or consists of a single prime p. In the latter case, $p = 2$ or $p \ge 5$. If $p \ge 5$, then $\sigma(N_2)/N_2 < 5/4$. If $p = 2$, then $\sigma(N_2)/N_2 \le \sigma(8)/8 = 15/8$, since $\nu_2(N) = \nu_2(D) \le 3$. Thus

$$\log \frac{\sigma(N_2)}{N_2} \le \log \frac{15}{8} < 0.63. \tag{39}$$

We now turn to the case $i = 3$. We claim that at most $V + K + 2$ primes in $\mathcal{A}_3$ divide U_m. Indeed, suppose that U_m is divisible by at least $V + K + 3$ such primes. Note that each such prime is at least 7. Since $\nu_2(N) \le V + 3$, at most $V + 3$ of these primes can appear with an odd exponent, so that at least K of them, say $R_1 < R_2 < \cdots < R_K$, must appear with an even exponent. Since $R_i^2 \mid U_m$, it follows that each $R_i \mid m$. Thus m is a multiple of the number

$$z(R_1) R_1 R_2 \cdots R_K.$$

But this number has at least $K + 1$ distinct prime divisors, whereas m has at most K distinct prime divisors. This contradiction proves the claim. Moreover, this argument shows that at most $K - 1$ of the primes in $\mathcal{A}_3$ dividing U_m appear to an exponent > 1 in the factorization of N. Consequently,

$$\log \frac{\sigma(N_3)}{N_3} \le \sum_{i=4}^{K+2} \log \left(1 + \frac{1}{p_i - 1}\right) + \sum_{i=K+3}^{V+K+5} \log \left(1 + \frac{1}{p_i}\right). \tag{40}$$

When $i = 4$, we have

$$\log \frac{\sigma(N_4)}{N_4} < \sum_{10^{10}<p<10^Z} \frac{1}{p-1}$$

$$< \sum_{10^{10}<p<10^Z} \frac{1}{p} + \sum_{n>10^{10}} \frac{1}{n(n-1)} = \sum_{10^{10}<p<10^Z} \frac{1}{p} + 10^{-10}.$$

Estimates (3.17) and (3.18) in [10] yield that

$$\sum_{10^{10}<p<10^Z} \frac{1}{p} < \log(0.1Z\log(10)) + \frac{1}{(10\log 10)^2}.$$

Collecting, we get

$$\log \frac{\sigma(N_4)}{N_4} < \log Z - 1.46. \tag{41}$$

We now take the case $i = 5$. We start off in a now-familiar way, noting that

$$\log \frac{\sigma(N_5)}{N_5} < \sum_{\substack{Q|U_m\\ Q\in\mathcal{A}_5}} \frac{1}{Q-1}. \tag{42}$$

The right-hand side of (42) is estimated by Abel summation. Let $A_5(t) = \#(\mathcal{A}_5 \cap [1,t])$ be the counting function of $\mathcal{A}_5$. Observe that

$$10^{ZA_5(t)} \le \prod_{\substack{p\in\mathcal{A}_5\\ p\le t}} p \le \prod_{2\le n<t^{2/5}} \Phi_n(10) \le 11^{\sum_{2\le n\le t^{2/5}}\varphi(n)},$$

so that

$$A_5(t) \le \left(\frac{\log 11}{Z\log 10}\right) \sum_{2\le n\le t^{2/5}} \varphi(n).$$

For each $x \ge 2$,

$$\sum_{2\le n\le x} \varphi(n) \le \sum_{1\le l\le \frac{x-1}{2}} \varphi(2l+1) + \sum_{1\le l\le \frac{x}{2}} \varphi(2l)$$

$$\le 2\sum_{1\le l\le \frac{x-1}{2}} l + \sum_{1\le l\le \frac{x}{2}} l \le \frac{3x^2}{8}.$$

Thus,

$$A_5(t) \le \left(\frac{3\log 11}{8Z\log 10}\right) t^{4/5} < 0.02t^{4/5} \qquad \text{for all} \quad Z \ge 20. \tag{43}$$

By Abel summation,

$$\sum_{\substack{p\in\mathcal{A}_5\\ p\le t}} \frac{1}{p-1} < \sum_{\substack{p\in\mathcal{A}_5\\ p\le t}} \frac{1}{p} + \sum_{n>10^{10}} \frac{1}{n(n-1)}$$

$$\le \frac{A_5(t)}{t} + \int_{10^Z}^{t} \frac{A_5(s)}{s^2} ds + 10^{-10}.$$

Now inserting (43), we find that

$$\sum_{\substack{p\in\mathcal{A}_5\\ p\le t}} \frac{1}{p-1} < 0.02\left(\frac{1}{t^{1/5}} + \int_{10^Z}^{t} \frac{ds}{s^{6/5}}\right) + 10^{-10} < \frac{5\times 0.02}{10^{Z/5}} + 10^{-10}.$$

Letting t go to infinity and recalling (42), we find that

$$\log\frac{\sigma(N_5)}{N_5} < \frac{1}{10^{1+Z/5}} + 10^{-10} \quad \text{whenever} \quad Z \ge 20. \tag{44}$$

Finally, we deal with the case $i = 6$. Suppose $Q \mid N_6$, and let $d = z(Q)$. Since $d^{5/2} \ge Q > 10^Z$, we have $d > 10^{2Z/5}$. Furthermore, $Q = dj + 1$ for some $j < d^{3/2}$. This last fact shows that

$$\sum_{\substack{Q\in\mathcal{A}_6\\ z(Q)=d}} \frac{1}{Q-1} \le \frac{1}{d}\sum_{j<d^{3/2}} \frac{1}{j} < \frac{1}{d}\left(1+\log(d^{3/2})\right) = \frac{3\log d}{2d}\left(1+\frac{2}{3\log d}\right).$$

Since $d > 10^{2Z/5}$, we have

$$\frac{3}{2}\left(1+\frac{2}{3\log d}\right) < \frac{3}{2}\left(1+\frac{2}{3\log(10^8)}\right) < 1.56 \quad \text{for} \quad Z \ge 20.$$

Thus, for each d,

$$\sum_{\substack{Q\in\mathcal{A}_6\\ z(Q)=d}} \frac{1}{Q-1} < \frac{1.56\log d}{d}. \tag{45}$$

It remains to sum over those d which appear as $z(Q)$ for some Q dividing N_6.

Since we are trying to lower the value of K, we may as well assume that K is a sharp upper bound and that N was chosen so that $k = K$. For each d as above, let $a_d = q_d^{\alpha_d}$ be the largest prime power divisor of d. Then

$$a_d \ge d^{1/K} > 10^{2Z/5K}.$$

Write $d = a_d e$ and group terms according to the value of a_d. Since $\log d \le K\log a_d$, we find that

$$\sum_{d} \frac{\log d}{d} \le \sum_{a_d > 10^{2Z/5K}} \frac{K\log a_d}{a_d} \sum_{\substack{e\mid m\\ \gcd(e,q_d)=1}} \frac{1}{e}.$$

Extending the inner sum to all positive integers e formed with prime factors $q \ne q_d$ of m, we get a sum whose value is

$$\prod_{\substack{q\mid m\\ q\ne q_d}} \left(1+\frac{1}{q-1}\right) = \left(\frac{m}{\varphi(m)}\right)\frac{q_d-1}{q_d}.$$

Thus,

$$\sum_d \frac{\log d}{d} \le K \frac{m}{\varphi(m)} \left\{ \sum_{\substack{q \mid m \\ q^b > 10^{2Z/5K}}} \frac{(q-1)\log(q^b)}{q^{b+1}} \right\}. \tag{46}$$

For each q dividing m, let β_q be the smallest positive integer satisfying $q^{\beta_q} > 10^{2Z/5K}$. Then

$$\sum_{\beta \ge \beta_q} \frac{(q-1)\log(q^\beta)}{q^{\beta+1}} = \frac{(q-1)\log(q^{\beta_q})}{q^{\beta_q+1}} \left(1 + \frac{1+1/\beta_q}{q} + \frac{1+2/\beta_q}{q^2} + \cdots\right)$$
$$\le \frac{(q-1)\log(q^{\beta_q})}{q^{\beta_q+1}} \left(1 + \frac{2}{q} + \frac{3}{q^2} + \ldots\right) = \frac{(q-1)\log(q^{\beta_q})}{q^{\beta_q+1}(1-1/q)^2} = \frac{q}{q-1}\frac{\log(q^{\beta_q})}{q^{\beta_q}}.$$

But $q^{\beta_q} > 10^{2Z/5K}$, and so

$$\sum_{\beta \ge \beta_q} \frac{(q-1)\log(q^\beta)}{q^{\beta+1}} < \left(\frac{2Z\log 10}{5K \cdot 10^{2Z/5K}}\right)\frac{q}{q-1}.$$

Summing over q and noting that $\frac{2\log 10}{5} < 0.922$, we obtain from (46) that

$$\sum_d \frac{\log d}{d} < \frac{0.922Z}{10^{2Z/5K}} \prod_{q \mid m}\left(1 + \frac{1}{q-1}\right)\left\{\sum_{q \mid m} \frac{q}{q-1}\right\}. \tag{47}$$

The right-hand side is largest when the K primes dividing m are $p_1, \ldots, p_K$. Using inequalities (45) and (47), we finally arrive at

$$\log \frac{\sigma(N_6)}{N_6} \le \sum_{\substack{Q \in \mathcal{A}_6 \\ Q \mid U_m}} \frac{1}{Q-1}$$
$$\le \left(\frac{1.44Z}{10^{2Z/5K}}\right)\prod_{i=1}^{K}\left(\frac{p_i}{p_i-1}\right)\left\{\sum_{i=1}^{K}\frac{p_i}{p_i-1}\right\}. \tag{48}$$

Combining estimates (38), (39), (40), (41), (44) and (48), we find that

$$\log \ell \le -0.42 + \sum_{i=4}^{K+2}\log\left(1 + \frac{1}{p_i-1}\right) + \sum_{i=K+3}^{V+K+5}\log\left(1+\frac{1}{p_i}\right) + \log Z$$
$$+ \frac{1}{10^{1+Z/5}} + \left(\frac{1.44Z}{10^{2Z/5K}}\right)\prod_{i=1}^{K}\left(\frac{p_i}{p_i-1}\right)\left\{\sum_{i=1}^{K}\frac{p_i}{p_i-1}\right\}. \tag{49}$$

We compare this with (31), which asserts when $k = K$ that

$$(2^{K+1} - 4K - 3)\log 2 < \log \ell. \tag{50}$$

Taking $K = 4$, $V = 13$, and $Z = 30$ in (49), we get $\log \ell < 4.85$, whereas the lower bound (50) gives $\log \ell > 6.23$. This contradiction shows that the case $k = 4$ is impossible, so that we may take $K = 3$.

Next, taking $K = 3$, $V = 13$ and $Z = 30$ in (49), we get $\log \ell \le 3.79$. Hence, we may take $L = 45$ and $V = \lfloor \log L / \log 2 \rfloor = 5$. Taking $K = 3$, $Z = 28$ and $V = 5$ in (49), we get $\log \ell \le 3.636$, therefore $\ell < 38$.

Thus, we may take $K = 3$ and $L = 37$.

4.3. The case when $3 \mid m$

Suppose that N is multiply perfect and $3 \mid m$. Then $3 \cdot 37 = U_3 \mid U_m \mid N$. Thus, either $37^2 \parallel N$ or $37 \parallel N = D \cdot U_m$. In the latter case,

$$19 \mid 38 = \sigma(37) \mid \sigma(N) = \ell N = \ell D \cdot U_m,$$

so that either $19 \mid \ell$ or $19 \mid U_m$. We will show that each of these possibilities leads to a contradiction.

Subcase: $37 \parallel N$ **and** $19 \mid \ell$. Since $\ell \le 37$, we must have $\ell = 19$. Also, $\nu_2(\sigma(N)) = \nu_2(19D) = \nu_2(D)$. If $D = 8$, then

$$5 \mid 15 = \sigma(8) \mid \sigma(8)\sigma(U_m) = \sigma(8U_m) = 19 \cdot 8 \cdot U_m,$$

so that $5 \mid U_m$, an absurdity. Suppose next that $D = 4$. Then

$$7 = \sigma(4) \mid \sigma(4U_m) = 19 \cdot 4 \cdot U_m,$$

so that $7 \mid U_m$ and hence $6 = z(7) \mid m$. Thus $U_6 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \mid U_m \mid N$. At most one of the four primes from $\mathcal{P} := \{7, 11, 13, 37\}$ can appear to higher than the first power in the prime factorization of N. Indeed, suppose $p \in \mathcal{P}$ is such that $p^2 \mid N = 4U_m$. Then $p^2 \mid U_m$, and so $p \mid z(p^2) \mid m$; but $\omega(m) = k \le 3$, and we know already that 2 and 3 divide m. It follows that

$$\nu_2(\sigma(N)) \ge 3;$$

but $\nu_2(\sigma(N)) = \nu_2(D\ell) = \nu_2(4 \cdot 19) = 2$. This contradiction establishes that $D \ne 4$. In the remaining cases, $\nu_2(\sigma(N)) = \nu_2(D) \le 1$. Thus, 37 is the only odd prime which appears to an odd power in the prime factorization of N. So the only primes which may appear to an odd power in $U_m = N/D$ are 37 and the odd prime divisors of D. So, either

$$U_m = 37 \cdot \Box, \quad \text{or} \quad U_m = 3 \cdot 37 \cdot \Box, \quad \text{or} \quad U_m = 7 \cdot 37 \cdot \Box.$$

The first and third possibilities can be immediately ruled out, since $U_m = \frac{10^m - 1}{9} \equiv -1 \pmod 8$ for $m \ge 3$, while an odd square is $\equiv 1 \pmod 8$. If $U_m = 3 \cdot 37 \cdot \Box$, then $3 \mid m$, and

$$\frac{(10^3)^{m/3} - 1}{10^3 - 1} = \frac{U_m}{U_3} = \Box.$$

Lemma 5 shows that this is impossible if $m/3 > 2$, which certainly holds in our case since $m \geq 70$.

Subcase: $37 \parallel N$ **and** $19 \mid U_m$. In this case, $2 \cdot 3^2 = 18 = z(19) \mid m$, so that

$$3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 52579 \cdot 333667 = U_{18} \mid U_m \mid N.$$

Let $\mathcal{P}$ be the set of primes > 3 which divide U_m. Since $k = \omega(m) \leq 3$, we see (arguing as in the earlier subcase) that at most one of the primes $p \in \mathcal{P}$ can divide U_m to a power higher than the first. Also, only the prime $p = 7$ in $\mathcal{P}$ can possibly divide D. It follows that at least $\#\mathcal{P} - 2 = 5$ of the primes in $\mathcal{P}$ exactly divide $N = D \cdot U_m$, and at least 3 of these belong to the residue class 3 mod 4. So

$$\nu_2(D\ell) = \nu_2(\sigma(N)) \geq 2 + 2 \cdot 3 = 8.$$

Since $\ell \leq 37$ and $D \leq 9$, this forces $D = 8$ and $\ell = 32$. But then $8 \parallel N$, so that $5 \mid 15 = \sigma(8) \mid \sigma(N) = 32 \cdot 8 \cdot U_m$, which implies that $5 \mid U_m$, an absurdity.

Subcase: $37^2 \mid N$. In this case, $3 \cdot 37 = z(37^2) \mid m$, so that $U_{3 \cdot 37} \mid U_m \mid N$. We first claim that either $2 \mid m$ or $3^2 \mid m$. Let $\mathcal{P}$ denote the set of primes $\neq 3, 37$ which divide $U_{3 \cdot 37}$. From the Cunningham project tables [15],

$$\begin{aligned}\mathcal{P} = \{&2028119, 247629013, 30557051518647307, 2212394296770203368013,\\ &8845981170865629119271997, 90077814396055017938257237117\}\end{aligned}$$

and each prime in $\mathcal{P}$ exactly divides $U_{3 \cdot 37}$. Since $k = \omega(m) \leq 3$, of the six primes in $\mathcal{P}$, at most one can divide U_m to a power higher than the first. Moreover, two of the primes in $\mathcal{P}$ belong to the progression 3 mod 4, and none of the primes in $\mathcal{P}$ can divide D. It follows that

$$\begin{aligned}\nu_2(D\ell) &= \nu_2(\sigma(N)) = \nu_2(\sigma(D \cdot U_m))\\ &\geq 4 + 2 \cdot 1 + \nu_2(\sigma(3^{\nu_3(N)})) = 6 + \delta, \end{aligned} \tag{51}$$

say. Since $\nu_2(D) \leq 3$, we have $\nu_2(\ell) \geq 3 + \delta \geq 3$, so that $\ell \in \{8, 16, 24, 32\}$. If $D = 8$, then $5 \mid \sigma(8) \mid \sigma(N) = 8\ell U_m$, which is impossible. If $D = 4$, then $7 = \sigma(4) \mid \sigma(N) = 8\ell U_m$, so that $6 = z(7) \mid m$; in particular, $2 \mid m$, and the claim holds in this case. In the remaining cases, $\nu_2(D) \leq 1$, so that (51) forces $\nu_2(\ell) \geq 5 + \delta$. Since $\ell \leq 37$, we must have $\ell = 32$, $\nu_2(\ell) = 5$, $\delta = 0$, and $\nu_2(D) = 1$. Since $\delta = 0$,

$$2 \mid \nu_3(N) = \nu_3(D) + \nu_3(U_m) = \nu_3(D) + \nu_3(m/3) + 1,$$

and so either $\nu_3(D) > 0$ or $3^2 \mid m$. In the latter case, we have the claim. If $3^2 \nmid m$, then $\nu_3(D) > 0$, which forces $D = 6$ (since $\nu_2(D) = 1$). Then $3^2 \parallel D \cdot U_m = N$, and

$$13 = \sigma(3^2) \mid \sigma(N) = \ell \cdot 6U_m.$$

Thus, $13 \mid U_m$, and so $6 = z(13) \mid m$. So the claim holds in this case too.

So either $2 \cdot 3 \cdot 37 \mid m$ or $3^2 \cdot 37 \mid m$. In the former case, $U_{2\cdot 3\cdot 37} \mid U_m \mid N$. Referring to the Cunningham tables, we see that there are 15 primes $p \notin \{3, 7, 37\}$ exactly dividing $U_{2\cdot 3\cdot 37}$. If $p^2 \mid U_m$ for any of these primes p, then $p \mid z(p^2) \mid m$, contradicting that $k = \omega(m) \leq 3$. Moreover, none of these primes p can divide D. So

$$\nu_2(D\ell) = \nu_2(\sigma(N)) = \nu_2(\sigma(D \cdot U_m)) \geq 15.$$

But $\nu_2(D\ell) \leq \nu_2(D) + \nu_2(\ell) \leq 3 + 5 = 8$, so this is absurd. In the latter case, another check in the Cunningham tables shows that there are 12 primes $p > 37$ exactly dividing $U_{3^2\cdot 37}$. At most one of these primes p can have $p^2 \mid U_m$, and so

$$\nu_2(D\ell) = \nu_2(\sigma(N)) = \nu_2(\sigma(D \cdot U_m)) \geq 11.$$

Again, this is impossible.

We conclude that $3 \nmid m$.

4.4. The case $k = 3$

Suppose now that $k = 3$. Since $3 \nmid m$, we have that D and U_m are relatively prime, so that $\sigma(N) = \sigma(D)\sigma(U_m)$. Since $3 \nmid m$, Lemma 8 shows that

$$\begin{aligned}\nu_2(D) + \nu_2(\ell) &= \nu_2(D\ell) = \nu_2(\sigma(N)) = \nu_2(\sigma(D \cdot U_m)) \\ &= \nu_2(\sigma(D)) + \nu_2(\sigma(U_m)) \geq \nu_2(\sigma(D)) + 2\omega''(U_m) \geq \nu_2(\sigma(D)) + 6.\end{aligned} \tag{52}$$

Since $\nu_2(D) \leq 3$, we get that $\ell \in \{8, 16, 24, 32\}$. As before, if $D = 8$, then $5 \mid \sigma(N) = 8\ell U_m$, which is impossible. Also, if $D = 4$, then $7 \mid 8\ell U_m$, so that $6 = z(7) \mid m$, contradicting that $3 \nmid m$. In the remaining cases, $\nu_2(D) \leq 1$ and

$$\nu_2(\ell) \geq (6 - \nu_2(D)) + \nu_2(\sigma(D)).$$

Since $\ell \leq 37$, this forces $\ell = 32$, $\nu_2(D) = 1$, and $\nu_2(\sigma(D)) = 0$. The only $D \in \{1, 2, \ldots, 9\}$ satisfying the last two conditions is $D = 2$. But if $D = 2$, then

$$3 = \sigma(2) \mid \sigma(N) = 32 \cdot (2U_m),$$

so that $3 \mid U_m$. But then $3 \mid m$, which we have seen is impossible.

We conclude that we may take $K = 2$.

4.5. Reduction to the repunit case ($D = 1$)

Suppose that $N = D \cdot U_m$ is multiply perfect. We shall show that U_m is also multiply perfect. We use that D and U_m are relatively prime to write

$$\sigma(D)\sigma(U_m) = \sigma(D \cdot U_m) = \ell D \cdot U_m. \tag{53}$$

Thus

$$\frac{\sigma(U_m)}{U_m} = \frac{\ell D}{\sigma(D)}.$$

We would like to show that $\sigma(D) \mid \ell D$. Since $\sigma(D) \mid \ell D \cdot U_m$ by (53), this follows if we show that $\sigma(D)$ is relatively prime to U_m. But this is easy: For each $D \in \{1, \dots, 9\}$, the number $\sigma(D)$ is supported on the set of primes $\mathcal{P} := \{2, 3, 5, 7, 13\}$. Clearly U_m is divisible by neither 2 nor 5. For the remaining primes $p \in \mathcal{P}$, we have $3 \mid z(p)$; since $3 \nmid m$, the number U_m is not divisible by these p either.

From now on we assume that $D = 1$, so that $N = U_m$.

4.6. Reduction to the case of odd m

At this point, it is convenient to compute a new bound on L, utilizing the reduced bound on K we obtained above. Taking $K = 2$, $Z = 20$, and $V = 5$ in (49), we find that $\log \ell < 3.17$, so that $\ell < 24$.

Now suppose that $N = U_m$ is multiply perfect and that $2 \mid m$. Then $11 = U_2 \mid U_m$. Write

$$U_m = U_2 \frac{U_m}{U_2}.$$

If $11 \nmid m$, then the right-hand factors are relatively prime, and

$$\ell U_m = \sigma(U_m) = \sigma(U_2)\sigma(U_m/U_2) = 12\sigma(U_m/U_2).$$

Since U_m is coprime to 12, it follows that $12 \mid \ell$, so that $\ell = 12$. Thus $\sigma(U_m/U_2) = U_m$. In particular, $\sigma(U_m/U_2)$ is odd. Hence, $U_m/U_2 = \square$, and so also $U_m U_2 = \square$. But $m > 2$, so that $U_m U_2 \equiv (-1)(3) \equiv 5 \pmod 8$, whereas an odd square is $\equiv 1 \pmod 8$. So it must be that $11 \mid m$. Thus, $U_{22} \mid U_m$. Now

$$U_{22} = 11^2 \cdot 23 \cdot 4093 \cdot 8779 \cdot 21649 \cdot 513239.$$

None of the fives primes $p > 11$ which U_{22} can divide U_m to a power higher than the first. Hence,

$$\nu_2(\ell) = \nu_2(\ell U_m) = \nu_2(\sigma(U_m)) \geq 5.$$

But this is impossible, since $\ell < 24$.

So, we may assume that m is odd.

4.7. Conclusion of the proof of Theorem 2

Suppose that U_m is multiply perfect. At this point, we know that m is coprime to both 2 and 3. Let $s := \Omega(m)$, and write $m = p_1 p_2 \cdots p_s$, where $p_1 \leq p_2 \leq \cdots \leq p_s$. Put $m_1 := m$, and for $1 \leq i \leq s$, successively define $m_{i+1} := m_i/p_i$. Then

$$U_m = \frac{U_{m_1}}{U_{m_2}} \frac{U_{m_2}}{U_{m_3}} \cdots \frac{U_{m_s}}{U_{m_{s+1}}}.$$

This decomposition of U_m is precisely the decomposition of U exhibited in the proof of Lemma 3. The argument given there shows that the greatest common divisor of any two of the s right-hand factors is supported on the primes dividing $g-1=9$. Since $3 \nmid m$, it follows that these s terms are pairwise coprime. By Lemma 5, none of the s terms is a square. It follows that the set of primes dividing U_m to an odd power has cardinality at least s. Moreover, since $U_m \equiv 3 \pmod 4$, at least one prime in this set belongs to the residue class $3 \bmod 4$. Hence,

$$\nu_2(\ell) = \nu_2(\sigma(U_m)) \geq s+1. \tag{54}$$

Since $\ell < 24$, we have $\nu_2(\ell) \leq 4$, and so $s \leq 3$.

We can now finish the demonstration of Theorem 2. With S_d defined as in (33), estimate (34) yields that

$$\begin{aligned} \log \ell = \log \frac{\sigma(U_m)}{U_m} \leq \sum_{Q \mid U_m} \frac{1}{Q-1} &\leq \sum_{\substack{d \mid m \\ d>1}} S_d \\ &\leq \sum_{\substack{d \mid m \\ d>1}} \frac{\log d}{d} \leq \frac{\log 5}{5} \#\{d \mid m : d > 1\}. \end{aligned}$$

On the other hand,

$$\log \ell \geq \nu_2(\ell) \log 2 \geq (s+1) \log 2.$$

Hence,

$$(s+1)\log 2 \leq \frac{\log 5}{5} \#\{d \mid m : d > 1\} \leq \frac{\log 5}{5}(2^s - 1).$$

But this inequality fails for each $s \leq 3$.

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