ON ORDERED FACTORIZATIONS INTO DISTINCT PARTS

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Abstract. Let $g(n)$ denote the number of ordered factorizations of n into integers larger than 1. In the 1930s, Kalmár and Hille investigated the average and maximal orders of $g(n)$. In this note we examine these questions for the function $G(n)$ counting ordered factorizations into *distinct* parts. Concerning the average of $G(n)$, we show that

$$\sum_{n\le x} G(n) = x \cdot L(x)^{1+o(1)},$$

where

$$L(x) = \exp\left(\log x \cdot \frac{\log\log\log x}{\log\log x}\right).$$

It follows that immediately that $G(n) \le n \cdot L(n)^{1+o(1)}$, as $n \to \infty$. We show that equality holds here on a sequence of n tending to infinity, so that $n \cdot L(n)^{1+o(1)}$ represents the maximal order of $G(n)$.

1. Introduction

Let $g(n)$ denote the number of factorizations of n into integers larger than 1, where factorizations with the same terms appearing in a different order are considered distinct. For example, $g(20) = 8$, corresponding to

$$20, \quad 4\cdot 5, \quad 5\cdot 4, \quad 2\cdot 10, \quad 10\cdot 2, \quad 2\cdot 2\cdot 5, \quad 2\cdot 5\cdot 2, \quad \text{and} \quad 5\cdot 2\cdot 2.$$

The study of statistical properties of $g(n)$ seems to have been initiated by Kalmár in the early 1930s. He proved [Kal32] that as $x \to \infty$,

$$\sum_{n\le x} g(n) = \frac{1}{-\zeta'(\rho)}\frac{x^\rho}{\rho} + o(x^\rho).$$

Here $\zeta(s)$ is the Riemann zeta function, and $\rho = 1.7286\ldots$ is the unique solution in $(1,\infty)$ to $\zeta(\rho) = 2$. For the size of the $o(x^\rho)$ error term, Kalmár obtained an upper bound of $O(x^\rho \exp(-c\log\log x \cdot \log\log\log x))$. This was improved by Ikehara [Ike41] to $O(x^\rho \exp(-c'(\log\log x)^{4/3-\epsilon}))$, and later by Hwang [Hwa00] to $O(x^\rho \exp(-c''(\log\log x)^{3/2-\epsilon}))$. In these estimates, $\epsilon > 0$ is arbitrary, and c, c', and c'' are positive constants (which may depend on ϵ).

In 1936, Hille [Hil36] took up the question of the maximal order of $g(n)$. He proved that $g(n) \ll n^\rho$, and that for every $\epsilon > 0$, there are infinitely many n with $g(n) > n^{\rho-\epsilon}$. Hille's results were refined by Erdős [Erd41] (who gave no proofs), Klazar–Luca [KL07] and Deléglise–Hernane–Nicolas [DHN08]. These last three authors prove that there are positive constants c and C such that

$$g(n) < n^\rho / \exp(c(\log n)^{1/\rho}/\log\log n)$$

for all large n, while

$$g(n) > n^\rho / \exp(C(\log n)^{1/\rho}/\log\log n)$$

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for infinitely many n. See also [CLM00], [CL05], and [BHT16].

In this note, we study the average and maximal order of the related function $G(n)$, which counts ordered factorizations of n into *distinct* parts larger than 1. (Thus, for instance, $G(20) = 5$.) Warlimont [War93] says that the study of $G(n)$ was suggested to him by A. Knopfmacher in private communication.

Warlimont writes (with notation changed to match ours): "At the time [when the problem was posed by Knopfmacher] it was not clear at all that $\sum_{n\le x} G(n) \ll x^{1+\epsilon}$." Warlimont (ibid.) subsequently proved that

$$\sum_{n\le x} G(n) \le x\cdot L(x)^{O(1)}, \tag{1}$$

where here and below

$$L(x) = \exp\left(\log x \frac{\log\log\log x}{\log\log x}\right).$$

This indeed shows that $\sum_{n\le x} G(n) \ll x^{1+\epsilon}$, so that $G(n)$ is considerably smaller on average than $g(n)$. Concerning (1), Warlimont comments: "I am still unable to prove a corresponding lower estimate for $\sum_{n\le x} G(n)$ ...."

Our first theorem determines the "correct" exponent of $L(x)$ in Warlimont's upper bound, while at the same time supplying a matching lower bound.

Theorem 1. *As $x \to \infty$,*

$$\sum_{n\le x} G(n) = x\cdot L(x)^{1+o(1)}.$$

An immediate consequence of Theorem 1 is that $G(n) \le n\cdot L(n)^{1+o(1)}$, as $n\to\infty$. We show that $n\cdot L(n)^{1+o(1)}$ is the true maximal order of $G(n)$.

Theorem 2. *There is a sequence of n tending to infinity along which*

$$G(n) \ge n\cdot L(n)^{1+o(1)}.$$

We conclude this introduction by mentioning that the analogous problems for unordered factorizations are already solved. Let $f(n)$ and $F(n)$ count unordered factorizations, with $F(n)$ carrying the restriction that the factors be distinct. An asymptotic formula for the average of $f(n)$ was obtained by Oppenheim [Opp27] and independently by Szekeres–Turán [ST33]. It is straightforward to modify their proofs to work for $F(n)$; doing so, one finds that $\sum_{n\le x} F(n) \sim \frac{1}{2}\sum_{n\le x} f(n)$, as $x\to\infty$. Thus, "on average" $F(n) \approx \frac{1}{2} f(n)$. (Cf. the discussion near the top of p. 180 of [Hen87].) Regarding maximal orders, it was proved in [CEP83] that both $f(n)$ and $F(n)$ have maximal order $n/L(n)^{1+o(1)}$.

2. Proof of Theorem 1

2.1. **Upper bound.** Our proof of the upper bound implicit in Theorem 1 is an elaboration on Warlimont's proof of (1). As in [War93], the idea is to apply "Rankin's trick." That is, we observe that

$$\sum_{n\le x} G(n) \le x^s \sum_{n=1}^{\infty} \frac{G(n)}{n^s}, \tag{2}$$

for any choice of $s > 1$, and we choose s to optimize the result.

Warlimont shows on pp. 189–191 of [War93] that for all $s > 1$,

$$\sum_{n=1}^{\infty} \frac{G(n)}{n^s} = \int_0^{\infty} e^{-t} \prod_{m>1} \left(1 + \frac{t}{m^s}\right) dt.$$

From this, he derives on p. 191 that (again, for $s > 1$)

$$\sum_{n=1}^{\infty} \frac{G(n)}{n^s} \leq 2 \cdot 3^{M(s)} \cdot (1 + \Gamma(M(s)+1)),$$

where

$$M(s) = \left\lfloor \exp\left(\frac{1}{s-1} \log \frac{2}{s-1}\right) \right\rfloor + 1.$$

From these last results and Stirling's formula, we find that as $s \downarrow 1$,

$$\sum_{n=1}^{\infty} \frac{G(n)}{n^s} \leq \exp\left(\exp\left((1+o(1))\frac{1}{s-1} \log \frac{1}{s-1}\right)\right).$$

With $\epsilon > 0$ arbitrary, choose s such that

$$s - 1 = (1+\epsilon)\frac{\log\log\log x}{\log\log x}.$$

We then deduce from (2) that for all large x,

$$\begin{aligned}\sum_{n \leq x} G(n) &\leq x \exp\left((s-1)\log x + \exp\left((1+o(1))\frac{1}{s-1}\log\frac{1}{s-1}\right)\right)\\ &\leq x \exp\left((1+2\epsilon)\log x \frac{\log\log\log x}{\log\log x}\right)\\ &= x \cdot L(x)^{1+2\epsilon}.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this establishes the upper bound implicit in Theorem 1.

Remark. The upper bound implicit in Theorem 1 can also be derived from a theorem of Mardjanichvili. Let $d_k(n)$ denote the number of expressions of n as an ordered product of k positive integers. Mardjanichvili proved [Mar39] that for all positive integers k, and all $x \geq 1$,

$$\sum_{n \leq x} d_k(n) \leq x \frac{(\log x + k - 1)^{k-1}}{(k-1)!}.$$

Now let K be the largest integer with $(K+1)! \leq x$, so that $K = (1+o(1)) \log x / \log\log x$ as $x \to \infty$. Observe that if $n \leq x$, then every ordered factorization of n into distinct integers larger than 1 involves at most K parts. Padding each factorization with 1s, we obtain an injection from the set counted by $G(n)$ into the set counted by $d_K(n)$. It follows that $\sum_{n \leq x} G(n) \leq x(\log x + K - 1)^{K-1}/(K-1)!$, once $x \geq 2$ (so that $K \geq 1$). A straightforward calculation with Stirling's formula shows that the upper bound here has size $x \cdot L(x)^{1+o(1)}$, as $x \to \infty$.

In some ways, this argument seems more flexible than Warlimont's. For instance, we can easily obtain the same upper bound if we relax the definition of $G(n)$ to allow factorizations with each part repeated at most L times, for any fixed L. This time we define K as the largest positive integer with $(\lfloor K/L \rfloor + 1)!^L \leq x$. This K still satisfies

$K = (1+o(1))\log x/\log\log x$ as $x \to \infty$, and the remainder of the argument goes through without change.

2.2. **Lower bound.** Fix $0 < \epsilon < 1$. For large x, let

$$y = (1-\epsilon)\frac{\log x}{\log\log x}, \quad \text{so that} \quad x^{1/y} = (\log x)^{1/(1-\epsilon)}.$$

Put

$$k = \lfloor y \rfloor - 1.$$

We consider (only) factorizations of the form $n_1 n_2 \cdots n_{k+1}$, where $n_1, n_2, \dots, n_k$ are distinct integers in $(1, x^{1/y}]$, and n_{k+1} is an integer in $(1, \frac{x}{n_1 \cdots n_k}]$ distinct from $n_1, \dots, n_k$. Clearly, $n_1 \cdots n_{k+1}$ is a factorization into distinct parts of a number in $[1, x]$, and so is counted in $\sum_{n \le x} G(n)$. Given $n_1, \dots, n_k$ as above,

$$\frac{x}{n_1 \cdots n_k} \ge x^{1-\frac{k}{y}} \ge x^{1/y} > 2(k+2).$$

Hence, the number of possible choices for n_{k+1} is

$$\left\lfloor \frac{x}{n_1 \cdots n_k} \right\rfloor - (k+1) \ge \frac{x}{n_1 \cdots n_k} - (k+2) \ge \frac{1}{2}\frac{x}{n_1 \cdots n_k}.$$

It follows that

$$\sum_{n \le x} G(n) \ge \frac{1}{2} x \sum_{\substack{n_1, \dots, n_k \in (1, x^{1/y}] \\ \text{distinct}}} \frac{1}{n_1 \cdots n_k}.$$

Given $n_1, \dots, n_{k-1} \in (1, x^{1/y}]$,

$$\sum_{\substack{n_k \in (1, x^{1/y}] \\ n_k \notin \{n_1, \dots, n_{k-1}\}}} \frac{1}{n_k} \ge \sum_{n \le x^{1/y}} \frac{1}{n} - \sum_{n=1}^{k} \frac{1}{n} \ge \log(x^{1/y}) - (1 + \log k),$$

and so

$$\sum_{\substack{n_1, \dots, n_k \in (1, x^{1/y}] \\ \text{distinct}}} \frac{1}{n_1 \cdots n_k} = \sum_{\substack{n_1, \dots, n_{k-1} \in (1, x^{1/y}] \\ \text{distinct}}} \frac{1}{n_1 \cdots n_{k-1}} \sum_{\substack{n_k \in (1, x^{1/y}] \\ n_k \notin \{n_1, \dots, n_{k-1}\}}} \frac{1}{n_k}$$
$$\ge (\log(x^{1/y}) - \log k - 1) \sum_{\substack{n_1, \dots, n_{k-1} \in (1, x^{1/y}] \\ \text{distinct}}} \frac{1}{n_1 \cdots n_{k-1}}.$$

Iterating, we are led to the lower bound

$$\sum_{\substack{n_1, \dots, n_k \in (1, x^{1/y}] \\ \text{distinct}}} \frac{1}{n_1 \cdots n_k} \ge (\log(x^{1/y}) - \log k - 1)^k.$$

With k and y as above,

$$\log(x^{1/y}) - \log k - 1 = \left(\frac{\epsilon}{1-\epsilon} + o(1)\right) \log\log x,$$

as $x \to \infty$. So for large x,

$$(\log(x^{1/y}) - \log k - 1)^k \ge \exp\left((1-2\epsilon)\log x \frac{\log\log\log x}{\log\log x}\right),$$

and

$$\sum_{n\le x} G(n) \ge x \exp\left((1-3\epsilon)\log x \frac{\log\log\log x}{\log\log x}\right) = x\cdot L(x)^{1-3\epsilon}.$$

This completes the proof of the lower bound.

3. Proof of Theorem 2

Recall that a positive integer n is called *z-smooth* if every prime factor of n belongs to the interval $[2,z]$. We follow convention in writing $\Psi(x,z)$ for the count of z-smooth integers in $[1,x]$. Below, a $'$ on a sum always indicates that the sum is to be restricted to integers that are $(\log x)$-smooth.

Theorem 2 is an easy consequence of the following estimate.

Lemma 3. *As $x\to\infty$,*

$$\sum_{n\le x}{}' \frac{G(n)}{n} \ge L(x)^{1+o(1)}.$$

Suppose Lemma 3 is proved. It is well-known (see, e.g., [Ten15, Theorem 5.2, p. 513]) that the count of $(\log x)$-smooth integers in $[1,x]$ is $\exp((2\log 2+o(1))\log x/\log\log x)$ as $x\to\infty$, and so in particular is $L(x)^{o(1)}$. So from Lemma 3, we may choose $n=n_x\in[1,x]$ such that

$$\frac{G(n)}{n}\ge L(x)^{1+o(1)}.$$

Clearly, $n\to\infty$ as $x\to\infty$. Since $n\le x$ and $L(x)$ is an increasing function, $L(x)\ge L(n)$, and

$$G(n)\ge n\cdot L(n)^{1+o(1)},$$

yielding Theorem 2.

It remains to prove Lemma 3.

Proof of Lemma 3. Fix a small $\epsilon>0$. For large x, let $y=(1-\epsilon)\log x/\log\log x$ (as before). We let $k=\lfloor y\rfloor$. If $n_1,\dots,n_k$ are distinct $(\log x)$-smooth integers in $(1,x^{1/y}]$, then $n_1 n_2\cdots n_k$ is a factorization into distinct parts of a $(\log x)$-smooth integer in $[1,x]$. Hence,

$$\sum_{n\le x}{}' \frac{G(n)}{n} \ge \sum_{\substack{n_1,\dots,n_k\in(1,x^{1/y}]\\ \text{distinct}}}{}' \frac{1}{n_1\cdots n_k}.$$

Reasoning as in the proof of Theorem 1, the right-hand side here has size at least

$$\left(\sum_{n\le x^{1/y}}{}' \frac{1}{n} - (1+\log k)\right)^k.$$

As $x\to\infty$,

$$\sum_{n\in(1,x^{1/y}]}{}' \frac{1}{n} = \sum_{1<n\le\log x}\frac{1}{n} + \sum_{\log x<n\le(\log x)^{1/(1-\epsilon)}}{}' \frac{1}{n}$$
$$= (1+o(1))\log\log x + \int_{\log x}^{(\log x)^{1/(1-\epsilon)}} \frac{d\Psi(t,\log x)}{t}.$$

We assume (as we may) that $\epsilon \le \frac{1}{2}$, so that $1/(1-\epsilon) \le 2$. As $x \to \infty$, we have, uniformly for t in our range of integration,

$$\begin{aligned}\Psi(t, \log x) &\ge \lfloor t \rfloor - \sum_{\log x < p \le t} \lfloor t/p \rfloor \\ &\ge t\left(1 - \sum_{\log x < p \le (\log x)^2} \frac{1}{p}\right) - 1 \\ &= t(1 - \log 2 + o(1)).\end{aligned}$$

Integrating by parts, it follows that

$$\int_{\log x}^{(\log x)^{1/(1-\epsilon)}} \frac{d\Psi(t, \log x)}{t} \ge (1 - \log 2 + o(1))\frac{\epsilon}{(1-\epsilon)} \log\log x.$$

Since $\log k = (1+o(1))\log\log x$, the above estimates combine to show that

$$\sum_{n \in (1, x^{1/y}]}{}' \frac{1}{n} - (1 + \log k) \ge (1 - \log 2 + o(1))\frac{\epsilon}{1-\epsilon} \log\log x.$$

Hence,

$$\begin{aligned}\sum_{n \le x}{}' \frac{G(n)}{n} &\ge \left(\sum_{n \le x^{1/y}}{}' \frac{1}{n} - (1 + \log k)\right)^k \\ &\ge \exp\left((1 - 2\epsilon)\log x \frac{\log\log\log x}{\log\log x}\right).\end{aligned}$$

Since ϵ can be arbitrarily small, the lemma follows. □

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