Perfection: A brief introduction

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Three kinds of natural numbers

Let $s(n) = \sum_{d|n, d<n} d$ be the sum of the proper divisors of n.

Abundant: $s(n) > n$.
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Perfect: $s(n) = n$.

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Perfect: $s(n) = n$.

For example, 5 is deficient ($s(5) = 1$, and similarly for any prime), 12 is abundant ($s(12) = 1 + 2 + 3 + 4 + 6 = 16$), and 6 is perfect ($s(6) = 1 + 2 + 3 = 6$).

You can see a lot just by looking

Abundants: 12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, ...

Deficients: 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, ...

Perfects: 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2658455991569831744654692615953842176, ...

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- *The limiting proportion of abundant numbers exists, is $\approx$ 24.76%.*
- *The limiting proportion of deficient numbers exists, is $\approx$75.24%.*

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Theorem

- *The limiting proportion of abundant numbers exists, is* $\approx$ *24.76%.*
- *The limiting proportion of deficient numbers exists, is* $\approx$*75.24%.*
- *The limiting proportion of perfect numbers exists, is exactly 0%.*

"Limiting proportion" = asymptotic density: Count up to x, divide by x, send $x \to \infty$.

The existence of these densities, as well as the fact that the perfect numbers have density 0, is a consequence of a general theorem proved by Davenport in early 1930s. (See book!)

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In this talk, we focus our attention on perfect numbers, which are arguably the most interesting of the three kinds.

Main theorems

Theorem (Euclid – Euler)

If $2^n - 1$ is prime, then

$$N := 2^{n-1}(2^n - 1)$$

is perfect. Conversely, if N is an even perfect number, then N has this form.

But what about odd perfect numbers?

Is there a simple formula for odd perfect numbers, like for even perfect numbers? Probably not.

Theorem (Dickson, 1913)

For each positive integer k, there are only finitely many odd perfect numbers with $\leq k$ distinct prime factors.

Dickson's theorem does not rule out the existence of odd perfect numbers. It is even compatible with their being infinitely many of them. But we can at least prove that it cannot be "too big" of an infinity.

Theorem (Hornfeck)

There are $\leq \sqrt{x}$ odd perfect numbers $\leq x$, for all $x \geq 1$.

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The rest of this talk is devoted to giving more or less complete proofs for these three theorems.

Euclid–Euler

Observe: N perfect $\iff \sigma(N) = 2N$.

Euclid's half of the theorem is "easy", given modern notations and notions. If $2^n - 1$ is prime, and

$$N := (2^n - 1)2^{n-1},$$

then

$$\sigma(N) = \sigma(2^n - 1)\sigma(2^{n-1})$$

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$$\begin{aligned} \sigma(N) &= \sigma(2^n - 1)\sigma(2^{n-1}) \\ &= 2^n \cdot (1 + 2 + 2^2 + \cdots + 2^{n-1}) \end{aligned}$$

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So N is perfect.

Define $h(N) = \frac{\sigma(N)}{N}$. So N is perfect $\iff h(N) = 2$.

Lemma

The function $h(N)$ is multiplicative. It is also "multiplicatively strictly increasing": If $a \mid b$, then $h(a) \leq h(b)$, with equality only if $a = b$.

Proof.

Multiplicativity: Inherited from $\sigma(N)$ and N.

Multiplicatively strictly increasing: Follows from the identity

$$h(N) = \sum_{d \mid N} \frac{1}{d}.$$

Lemma

Let m be an integer > 1. If $h(N) = \frac{m+1}{m}$, then m is prime, and $N = m$.

Proof.

Write $\frac{\sigma(N)}{N} = \frac{m+1}{m}$. RHS is in lowest terms. Thus, $m \mid N$.

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Equality holds throughout. We need $\sigma(m) = m + 1$, so m is prime.
And we need $h(N) = h(m)$, so $N = m$.

Proof that if N is even perfect, N has Euler's form.

Write $N = 2^k Q$, where Q is odd. Starting from $\sigma(N) = 2N$, get

$$\begin{aligned} 2^{k+1} Q &= \sigma(2^k)\sigma(Q) \\ &= (2^{k+1} - 1)\sigma(Q). \end{aligned}$$

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Rearrange: $h(Q) = \frac{\sigma(Q)}{Q} = \frac{2^{k+1}}{2^{k+1}-1} = \frac{m+1}{m}$ where

$$m = 2^{k+1} - 1.$$

By lemma: $m = 2^{k+1} - 1$ is prime, and $Q = m = 2^{k+1} - 1$.

So $N = 2^k(2^{k+1} - 1)$.

Dickson's finiteness theorem

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We will give a **supernatural proof** of this theorem, due (essentially) to HN Shapiro.

Supernatural numbers

Definition

A **supernatural number** is a formal product

$$2^{e_2} 3^{e_3} 5^{e_5} \cdots = \prod_{p \text{ prime}} p^{e_p},$$

where each $e_p \in \{0, 1, 2, 3, \ldots\} \cup \{\infty\}$.

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2. $2 \cdot 3^{\infty} \cdot 17$ is also a supernatural number, as is $\prod_p p^{\infty}$.

There is a natural notion of what it means for one supernatural number to divide another.

Definition (p-adic valuation)

If N is a supernatural number, and p is a prime, we let $v_p(N)$ be the exponent of p in the factorization of N. Thus, $v_p(N) \in \{0, 1, 2, \dots\} \cup \{\infty\}$.

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Definition (supernatural convergence)

If $N_1, N_2, N_3, \ldots$ is a sequence of supernatural numbers, and N is a supernatural number, we say $N_i \to N$ if:

For every prime p, we have $v_p(N_i) \to v_p(N)$.

Examples

- $2, 3, 5, 7, 11, 13, \ldots$ converges to 1.
- $2, 2^2 \cdot 3^2, 2^3 \cdot 3^3 \cdot 5^3, \ldots$ converges to $\prod_p p^\infty$.

Lemma

Every sequence of supernatural numbers has a subsequence that converges to a supernatural number.

Proof.

Exercise! (Related to Tychonoff's theorem.)

For each positive integer k, let $\mathcal{S}_k$ be the set of supernatural numbers where at most k exponents are nonzero.

Lemma

If $N_1, N_2, N_3, \ldots$ is a sequence of elements of $\mathcal{S}_k$ converging supernaturally to a limit N. Then $N \in \mathcal{S}_k$.

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Proof.

If not, there are at least $k+1$ different primes p for which $v_p(N) \neq 0$.

Let p be one of these primes. By definition, $v_p(N_i) \to v_p(N)$.

If $v_p(N) < \infty$, then $v_p(N_i) = v_p(N)$ for all large i. If $v_p(N) = \infty$, then $v_p(N_i)$ is eventually arbitrarily large. In either case, $v_p(N_i)$ is nonzero for all large i.

But then N_i has at least $k+1$ nonzero exponents eventually. This contradicts that each $N_i \in \mathcal{S}_k$.

Lemma

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Proof.

Let p be a prime dividing N. Say $v_p(N) = e_p$. By definition, $v_p(N_i) = e_p$ for all large i. Choose i large enough that this holds simultaneously for all the (finitely) many primes p dividing N.

Then for all large i, we have $v_p(N) \leq v_p(N_i)$ for all primes p. So $N \mid N_i$.

Recall that $h(N) = \frac{\sigma(N)}{N}$. We can extend $h(N)$ to S_k. How?
If $N \in \mathcal{S}_k$, define

$$h(N) = \prod_p h(p^{e_p}).$$

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This is "morally" a finite product.

Here we understand

$$h(p^\infty) = \lim_{e\to\infty} h(p^e) = \lim_{e\to\infty} \frac{(p^{e+1}-1)/(p-1)}{p^e} = \frac{p}{p-1}.$$

If N is a natural number with $\leq k$ prime factors, then $h(N)$ makes sense with N thought of as either a natural number, or an element of $\mathcal{S}_k$, and we get the same answer.

Lemma (Continuity lemma)

If $N_1, N_2, N_3, \ldots$ is a sequence of elements of S_k converging supernaturally to N, then $h(N_i) \to h(N)$.

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Proof.

Write $N = p_1^{e_1} \cdots p_\ell^{e_\ell}$, where $e_1, \ldots, e_\ell$ are the nonzero exponents in the factorization of N. We can write

$$N_i = p_1^{e_{1,i}} p_2^{e_{2,i}} \cdots p_\ell^{e_{\ell,i}} M_i,$$

where M_i is the part of the factorization consisting of primes other than $p_1, \ldots, p_\ell$. Then

$$h(N_i) = h(p_1^{e_{1,i}}) h(p_2^{e_{2,i}}) \cdots h(p_\ell^{e_{\ell,i}}) h(M_i).$$

And as $i \to \infty$,

$$h(p_1^{e_{1,i}}) h(p_2^{e_{2,i}}) \cdots h(p_\ell^{e_{\ell,i}}) \to h(N).$$

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$$N_i = p_1^{e_{1,i}} p_2^{e_{2,i}} \cdots p_\ell^{e_{\ell,i}} M_i.$$

Observe: For every p, the exponent $v_p(M_i)$ is eventually zero.

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Observe: For every p, the exponent $v_p(M_i)$ is eventually zero.

Claim: $h(M_i) \to 1$.

When $M_i = 1$, also $h(M_i) = 1$. For other i, let $q_i =$ least prime with a nonzero exponent in M_i. Then $h(M_i)$ is a product of at most k numbers, each of the form $h(q^e)$ with $q \geq q_i$. It follows that

$$1 \leq h(M_i) \leq \left(\frac{q_i}{q_i - 1}\right)^k.$$

But $q_i \to \infty$ along the sequence of M_i for which q_i exists, so upper bound $\to 1$. Completes the proof of continuity lemma.

Proof of Dickson's theorem.

Suppose for a contradiction that there are infinitely many odd perfect numbers with $\leq k$ distinct prime factors.

Then we can choose a supernaturally convergent sequence of distinct such numbers, say $N_1, N_2, N_3, \dots$. Say $N_i \to N$, where

$$N = p_1^{e_1} \cdots p_r^{e_r},$$

where $r \leq k$.

Each $h(N_i) = 2$, so $h(N) = \lim h(N_i) = 2$.

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where $r \leq k$.

Each $h(N_i) = 2$, so $h(N) = \lim h(N_i) = 2$.

Observation: At least one of the exponents $e_j = \infty$. Otherwise, N is a natural number, and N divides N_i for all large i. At most one N_i can equal N. So from some point on, N is a proper divisor of N_i, meaning that

$$2 = h(N_i) > h(N) = 2.$$

Write

$$N = p_1^{e_1} \cdots p_r^{e_r},$$

where $r \leq k$ and $h(N) = 2$.

Can order the primes so that $e_1, \ldots, e_\ell < \infty$, and $e_{\ell+1}, \ldots, e_r = \infty$. Then

$$2 = \frac{p_1^{e_1+1} - 1}{p_1^{e_1}(p_1 - 1)} \cdots \frac{p_\ell^{e_\ell+1} - 1}{p_\ell^{e_\ell}(p_\ell - 1)} \cdot \frac{p_{\ell+1}}{p_{\ell+1} - 1} \cdots \frac{p_r}{p_r - 1}.$$

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Clear some denominators:

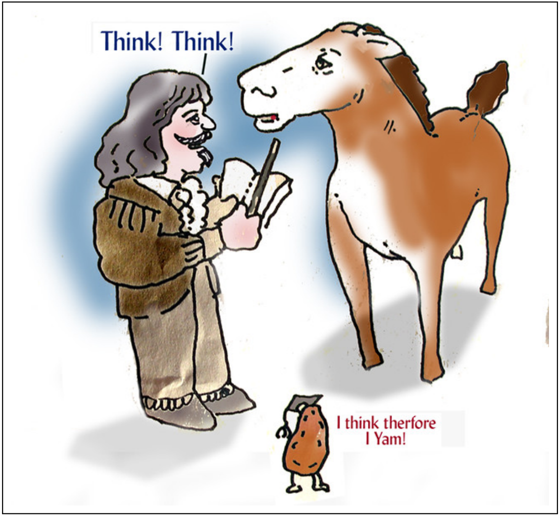
$$2p_1^{e_1} \cdots p_\ell^{e_\ell}(p_{\ell+1} - 1) \cdots (p_r - 1) = \frac{p_1^{e_1+1} - 1}{p_1 - 1} \cdots \frac{p_\ell^{e_\ell+1} - 1}{p_\ell - 1} \cdot p_{\ell+1} \cdots p_r.$$

Can assume $p_{\ell+1} < \cdots < p_r$. Then p_r divides RHS but not LHS !

Hornfeck's theorem

Theorem (Hornfeck)

There are $\leq \sqrt{x}$ *odd perfect numbers* $\leq x$.



Lemma (Descartes)

If N is odd and perfect, then $N = p^k m^2$, where the prime p does not divide m, and k is odd.

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Proof.

If q is an odd prime, then

$$\begin{aligned} \sigma(q^e) &= 1 + q + \dots + q^e \\ &\equiv e + 1 \pmod 2. \end{aligned}$$

So $\sigma(q^e)$ is odd if and only if e is even.

Now write $N = \prod_{q^e \| N} q^e$. Then $2N = \prod_{q^e \| N} \sigma(q^e)$. Since $2N$ is twice an odd number, each $\sigma(q^e)$ is odd, with one exception.

Proof of Hornfeck's theorem.

Let N be odd perfect, $N \leq x$, and write $N = p^k m^2$ a la Descartes. We will show each m corresponds to at most one N. Since $m \leq \sqrt{x}$, result will follow.

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Since $2N = \sigma(N)$, we get $2p^k m^2 = \sigma(p^k)\sigma(m^2)$, and so

$$\begin{aligned} 2\sigma(m^2)/m^2 &= \sigma(p^k)/p^k \\ &= (1 + p + \cdots + p^k)/p^k. \end{aligned}$$

RHS is in lowest terms, LHS depends only on m.

Hence, p^k is determined by m: It is the denomintor when LHS is put in lowest terms!

Where do we stand today?

We still do not know if there are infinitely many even perfect numbers, because we do not know if there are infinitely many primes of the form $2^n - 1$.

It is easy to see we only need to consider $2^p - 1$, with p itself prime.

But we can prove almost nothing about numbers of the form $2^p - 1$. We cannot even show $2^p - 1$ is composite infinitely often!

Where do we stand today?

After Heath-Brown, Cook, and Nielsen, we have the following explicit forms of Dickson's theorem.

Theorem

If N is odd and perfect with $\leq k$ distinct prime factors, then $N < 2^{4^k}$.

As a complement to this:

Theorem (P.)

The number of odd perfect N with $\leq k$ distinct prime factors is $< 4^{k^2}$.

Theorem (Hornfeck–Wirsing)

For each $\epsilon > 0$, the number of odd perfect $N \leq x$ is $< x^{\epsilon}$, for all $x > x_0(\epsilon)$.

Theorem (Wirsing)

For some absolute constant C and all large x, the number of odd perfect $N \leq x$ is at most $x^{C/\log\log x}$.

Wirsing's theorem (1959) is still the "state-of-the-art": After 60+ years, we still do not know how to show that C can be taken arbitrarily small in that result.