The quest for perfection

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WHAT IS... a perfect number?

Let $\sigma(n) := \sum_{d|n} d$ be the usual sum-of-divisors function, and let $s(n) := \sum_{d|n, d<n} d$ be the sum-of-proper-divisors function, so that $s(n) = \sigma(n) - n$.

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A natural number n is called **perfect** if $\sigma(n) = 2n$, or equivalently, if $s(n) = n$.

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But who decided adding divisors was a reasonable thing to do in the first place?

All Greek to us

Among simple even numbers, some are superabundant, others are deficient: these two classes are as two extremes opposed one to the other; as for those that occupy the middle point between the two, they are said to be perfect.

– Nicomachus (ca. 100 AD), *Introductio Arithmetica*

Abundant: $s(n) > n$, e.g., $n = 12$.
Deficient: $s(n) < n$, e.g., $n = 5$.
Perfect: $s(n) = n$, e.g., $n = 6$.

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Carl Pomerance has called this the "Goldilox classification".

Goldilox explained

The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.

. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.

You can see a lot just by looking

Let's list the first several terms of each of these sequences.

Abundants: 12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, ...

Deficients: 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, ...

Perfects: 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, ...

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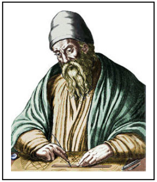
> *Just as . . . ugly and vile things abound, so superabundant and deficient numbers are plentiful and can be found without a rule. . . – Nicomachus*

Here's looking at Euclid

A rule for generating perfect numbers was given by Euclid in his *Elements*, written around 300 BCE.



Theorem (Euclid)

If as many numbers as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes a prime, and if the sum multiplied into the last make some number, the product will be perfect.

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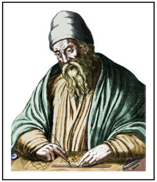

Theorem (Euclid)

If $2^n - 1$ *is a prime number, then*

$$N := 2^{n-1}(2^n - 1)$$

is a perfect number.

For example: $N = 6$ $(n = 2)$, $N = 28$ $(n = 3)$, and

$N = 8116868 \cdots 22457856$ corresponding to $n = 3021377$.

(About 1.8 million middle digits omitted.)

A sequel two thousand years in the making

Theorem (Euler)

If N is an even perfect number, then N can be written in the form

$$N = 2^{n-1}(2^n - 1),$$

where $2^n - 1$ is a prime number.

There are several proofs of Euler's theorem known. We choose to present an elegant argument of Graeme Cohen from the *Math. Gazette*.

The mouth speaketh

Let's define the **abundancy** of the number n as the ratio

$$h(n) := \sigma(n)/n.$$

We use the following simple lemma:

Lemma

If $n \mid m$, then $h(n) \leq h(m)$. Equality holds only if $n = m$.

Proof.

This is immediate upon observing that for every n,

$$h(n) = \frac{1}{n}\sum_{d|n} d = \sum_{d|n}\frac{d}{n} = \sum_{d|n}\frac{1}{n/d} = \sum_{e|n}\frac{1}{e}.$$

Making Euler slicker

We now prove Euler's characterization of even perfect numbers, following Cohen. Say N is even perfect. Write $N = 2^k q$, where q is odd and $k \geq 1$.

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It follows that $2^{k+1} - 1$ divides q.

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It follows that $2^{k+1}-1$ divides q. Hence,

$$\begin{aligned} 2 = \frac{\sigma(N)}{N} &= \frac{2^{k+1}-1}{2^k} \cdot \frac{\sigma(q)}{q} \\ &\geq \frac{2^{k+1}-1}{2^k} \cdot \frac{\sigma(2^{k+1}-1)}{2^{k+1}-1} \geq \frac{2^{k+1}-1}{2^k} \cdot \frac{2^{k+1}}{2^{k+1}-1} = 2. \end{aligned}$$

Making Euler slicker

We've just seen that

$$2 = \frac{\sigma(N)}{N} = \frac{2^{k+1}-1}{2^k} \cdot \frac{\sigma(q)}{q}$$
$$\geq \frac{2^{k+1}-1}{2^k} \cdot \frac{\sigma(2^{k+1}-1)}{2^{k+1}-1} \geq \frac{2^{k+1}-1}{2^k} \cdot \frac{2^{k+1}}{2^{k+1}-1} = 2.$$

So equality holds throughout. This means that

$$2^{k+1} - 1 = q$$

and that

$$\sigma(2^{k+1}-1) = 2^{k+1},$$

which forces $2^{k+1}-1$ to be prime. So $N = 2^k(2^{k+1}-1)$, where the second factor is prime.

The media's anti-Platonist bias

So even perfect numbers are in one-to-one correspondence with Mersenne primes. Great!

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But we don't know if there are infinitely many Mersenne primes, though there is some good evidence for this.

The Other 49 Best Inventions

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29. The 46th Mersenne Prime

A Mersenne number is a positive number that can be expressed in the form 2n-1. A Mersenne prime is a Mersenne number that is, well, prime. Searching for higher and higher Mersenne primes is the unofficial national sport of mathematicians. The 45th and 46th (right) Mersenne primes were found this year, the latter by a team at UCLA. It has almost 13 million digits.

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We don't even have a proof that there are infinitely *composite* numbers of the form $2^p - 1$.

Odd perfect numbers: the (probably non-existent) elephant in the room

Maybe the oldest open problem in number theory is the question of whether there exist odd perfect numbers.

Is there a formula for odd perfect numbers, like the Euclid–Euler formula? Probably not.

Theorem (Dickson, 1913)

For each positive integer k, there are only finitely many odd perfect numbers N with precisely k distinct prime factors.

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How many is finitely many? Well,

$$N < (4k)^{(4k)^{2^{k^2}}} \qquad \text{(Pomerance, 1977).}$$

Size matters

Pomerance's result was later refined by Heath-Brown, Cook, and finally Nielsen:

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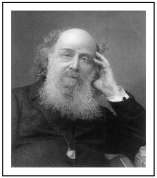
Theorem (P.)

The number of odd perfect N with k distinct prime factors is at most

$$4^{k^2}.$$

Sylvester's web

. . . a prolonged meditation has satisfied me that the existence of [an odd perfect number] - its escape, so to say, from the complex web of conditions which hem it in on all sides - would be little short of a miracle. – J. J. Sylvester



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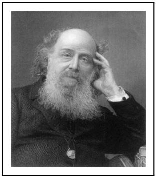

We know quite a few conditions an odd perfect number must satisfy. For instance:

1. N has the form $p^e M^2$, where $p \equiv e \equiv 1 \pmod 4$ (Euler),
2. N has at least 9 distinct prime factors (Nielsen)
3. N has at least 101 prime factors counted with multiplicity (Ochem and Rao),
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Does this justify Sylvester's conjecture that odd perfect numbers don't exist? It's amusing to note that almost all natural numbers satisfy these four conditions!

A new hope

Rather than try to study the individual members of a set, one can take a **statistical perspective**, examining how many elements of the set there are below a given bound.

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If A is a subset of $\mathbb{N} = \{1, 2, 3, \dots\}$, define the *density* of A as

$$\lim_{x\to\infty} \frac{\#A \cap [1, x]}{x}.$$

For example, the even numbers have density $1/2$, and the prime numbers have density 0. But the set of natural numbers with first digit 1 does not have a density.

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Question

Does the set of abundant numbers have a density? What about the deficient numbers? The perfect numbers?

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Theorem (Davenport, 1933)

For each real $u \geq 0$, consider the set

$$\mathcal{D}_s(u) = \{n : s(n)/n \leq u\}.$$

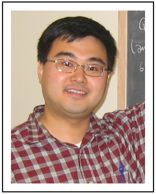
This set always possesses an asymptotic density $D_s(u)$. Considered as a function of u, the function D_s is continuous and strictly increasing, with $D_s(0) = 0$ and $D_s(\infty) = 1$.

Corollary

The perfect numbers have density 0, the deficient numbers have density $D_s(1)$, and the abundant numbers have density $1 - D_s(1)$.

Numerics

The following theorem improves on earlier work of Behrend, Salié, Wall, and Deléglise:



Theorem (Kobayashi, 2010)

For the density of abundant numbers, we have

$$0.24761 < 1 - D_s(1) < 0.24765.$$

So just under 1 in every 4 natural numbers is abundant, and just over 3 in 4 are deficient.

Counting perfect numbers

Let $V(x)$ denote the number of perfect numbers $n \leq x$. Davenport's theorem says that $V(x)/x \to 0$ as $x \to \infty$. Can we say anything more precise?

Even perfect numbers correspond to primes of the form $2^n - 1$. We know 47 such values of n, the largest being

$$n = 42643801.$$

Conjecture

The number of $n \leq x$ for which $2^n - 1$ is prime is asymptotic to

$$\frac{e^\gamma}{\log 2} \log x.$$

An exercise in heuristic reasoning

Where does this conjecture come from? First, if $2^n - 1$ is prime, then $n = p$ is also prime. A random number of size $\approx 2^p$ is prime with probability about $\frac{1}{\log(2^p)} = \frac{1}{p\log 2}$. But $2^p - 1$ has a 'leg up' on being prime: One can easily prove that it's never divisible by any prime $< 2p$. This suggests multiply our naive probability by

$$\prod_{q<2p} (1 - 1/q)^{-1} \approx e^{\gamma} \log p.$$

This gives a "corrected" probability of about $e^{\gamma}/\log 2 \cdot \log p/p$.

Summing this probability over $p \leq x$ gives us an expected count of about $\frac{e^{\gamma}}{\log 2} \log x$.

An exercise in heuristic reasoning

This conjecture on the distribution of Mersenne primes suggests a corresponding conjecture for even perfect numbers:

Conjecture

The number of even perfect numbers up to x is asymptotic to $\frac{e^\gamma}{\log 2}\log\log x$, as $x \to \infty$.

Since we don't think there are any odd perfect numbers, we can also erase the word "even".

A good half-decade for perfect numbers

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Hornfeck & Wirsing, 1957	$V(x) = O(x^{\epsilon})$

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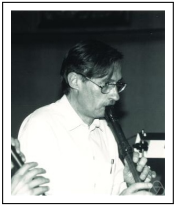

The sharpest result to date is due to Wirsing (1959): For all $x > 3$,

$$V(x) \leq x^{W/\log\log x}$$

for a certain absolute constant W. This is no doubt still very far from the truth.

In the opposite direction, the following conjecture is wide open:

Conjecture

There are infinitely many perfect numbers, i.e., $V(x) \to \infty$ as $x \to \infty$.

Higher order generalizations of the perfect numbers

Problem (Catalan, 1888)

Start with a natural number n. What is the eventual behavior of the sequence of iterates $n, s(n), s(s(n)), s(s(s(n))), \ldots$ (the aliquot sequence at n)?

Example

$n = 20$ leads to the sequence $20, 22, 14, 10, 8, 7, 1, 0$.

Example

$n = 25$ leads to the sequence $25, 6, 6, 6, 6, 6, \ldots$.

Higher order generalizations

Conjecture (Catalan)

Every starting n leads to a sequence terminating at 0, or hits a perfect number.

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As noticed almost immediately by Perrott, this is false:

Example

$n = 220$ leads to the sequence $220, 284, 220, 284, 220, 284, \ldots$.

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Every starting n leads to a sequence terminating at 0, or hits a perfect number.

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Example

$n = 220$ leads to the sequence $220, 284, 220, 284, 220, 284, \dots$.

Conjecture (Catalan–Dickson, 1913)

Every starting n leads to a bounded sequence, i.e., either a sequence terminating in 0 or reaching a cycle.

The first n for which this conjecture is in doubt is $n = 276$.

Higher order generalizations

Definition

We call n a *sociable number* if the aliquot sequence at n is purely periodic; in this case, the length of the period is called the *order* of sociability.

Definition

An *amicable number* is a sociable number of order 2. In this case, the pair $\{n, s(n)\}$ is an amicable pair.

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Persons who have concerned themselves with talismans affirm that the amicable numbers 220 and 284 have an influence to establish a union or close friendship between two individuals. – Ibn Khaldun

Al-Majriti (ca. 1050 years ago) claims to have tested the erotic effect of

> *giving any one the smaller number 220 to eat, and himself eating the larger number 284.*

The distribution of amicable numbers

Let $V_k(x)$ denote the number of sociable numbers of order k not exceeding x. (So $V(x) = V_1(x)$.)

Conjecture (Bratley, Lunnon, and McKay)

$V_2(x)/x^{1/2} \to 0$ as $x \to \infty$.

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For each $\epsilon > 0$, we have $V_2(x) > x^{1-\epsilon}$ once $x > x_0(\epsilon)$.

Though we know > 12 million amicable pairs, we still have no proof that there are infinitely many, or even any plausible strategy.

The distribution of amicable numbers

The first theorem in this area is due to Erdős.

Theorem (Erdős 1955)

The set of natural numbers which belong to an amicable pair has asymptotic density zero.

A more explicit result was obtained in joint work with Rieger twenty years later.

Theorem (Erdős and Rieger 1975)

The set of amicable numbers has asymptotic density zero. In fact,

$$V_2(x) \ll \frac{x}{\log\log\log x}.$$

A wholly inadequate sketch of the proof that $V_2(x)/x \to 0$

It's enough to prove this for the set consisting of the *smaller* member of each amicable pair.

Applying s to any of the numbers n in this set, we jump to the larger member $s(n)$ of the pair. So each n in this set is abundant. But if we apply s one more time, we jump back down to n. So $s(n)$ is deficient.

What Erdős really proved was the following theorem:

Theorem

If n is abundant, then almost all of the time, $s(n)$ is also abundant. More precisely, the set of abundant n for which $s(n)$ is non-abundant has density zero.

A wholly inadequate sketch of the proof that $V_2(x)/x \to 0$

How do you prove the theorem? Notice that whether or not n is abundant depends on the size of the ratio

$$\frac{\sigma(n)}{n} = \prod_{p^e \| n} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^e}\right),$$

and the size of this ratio is usually governed by the small prime powers dividing n.

Erdős's idea was to show that almost always, $s(n)$ and n have the same set of small prime power divisors. He argues that if n is abundant, it's probably the small prime powers that put it over the top, and since $s(n)$ shares these small prime powers, $s(n)$ should be abundant too!

Sketchiness, continued

Why should n and $s(n)$ share the same small prime power factors? By definition,

$$s(n) = \sigma(n) - n.$$

Erdős noticed that $\sigma(n)$ is usually divisible by high powers of all the small primes (powers higher than those that appear in a typical number n). So the ultrametric inequality forces $s(n)$ and n to share the same small prime power divisors.

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Why should $\sigma(n)$ be so highly divisible? Let's start by looking at a special case: Why is $\sigma(n)$ almost always divisible by 3?

Sketchiness, continued

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A classical result of Hardy and Ramanujan states that almost all numbers n have roughly $\log\log n$ distinct prime factors; moreover, this holds even if we count only primes that appear only to the first power.

Sketchiness, continued

A classical result of Hardy and Ramanujan states that almost all numbers n have roughly $\log\log n$ distinct prime factors; moreover, this holds even if we count only primes that appear only to the first power.

If q is a prime that appears to the first power in the factorization of n, then

$$q+1=\sigma(q)\mid\sigma(n).$$

So if $3\mid q+1$, then $3\mid\sigma(n)$.

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$$q + 1 = \sigma(q) \mid \sigma(n).$$

So if $3 \mid q + 1$, then $3 \mid \sigma(n)$.

We said above that there are $\log\log n$ of these primes q (typically). Half of the primes are $-1 \pmod 3$, so it's highly unlikely that *none of our* q satisfy $q \equiv -1 \pmod 3$.

This (non-rigorous) argument suggests that $\sigma(n)$ is, almost all of the time, divisible by every natural number up to about $\log\log n$. One can prove a (slightly weaker) version of this claim using sieve methods.

The distribution of amicable numbers

Pomerance proved a much sharper upper bound on $V_2(x)$ than Erdős and Rieger.

Theorem (Pomerance, 1981)

For all large x,

$$V_2(x) \le \frac{x}{\exp((\log x)^{1/3})}.$$

As a consequence, the sum of the reciprocals of the amicable numbers converges.

Sociable numbers of higher order, or "three is a crowd"

Example

Here is a sociable cycle of order 5 (found by Poulet in 1918):

$$12496 \to 14288 \to 15472 \to 14536 \to 14264 \to 12496 \to \dots$$

order of the cycle	number of known examples
1	47
2	>12 million
4	206
5	1
6	5
8	3
9	1
28	1

The distribution of sociable numbers

Theorem (Erdős, 1976)

For each fixed k, the set of sociable numbers of order k has density zero.

Erdős's bounds for $V_k(x)$ were very weak. How weak? His method proves:

The distribution of sociable numbers

Theorem (Erdős, 1976)

For each fixed k, the set of sociable numbers of order k has density zero.

Erdős's bounds for $V_k(x)$ were very weak. How weak? His method proves:

$$V_k(x) \ll \frac{x}{\log\log\log\log\cdots\log x},$$

where the denominator is a $(3k)$-fold iterated $\log$.

With Kobayashi and Pomerance, we improved this. For example, we can replace the denominator with any fixed power of $\log_3 x$.

All together now

Put

$$V^*(x) = V_1(x) + V_2(x) + V_3(x) + \dots,$$

so that $V^*(x)$ is the counting function of *all* the sociable numbers.

Conjecture

As $x \to \infty$, we have $V^*(x)/x \to 0$. In other words, the set of sociable numbers has density zero.

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Theorem (Kobayashi, Pomerance, P., 2009)

The set of deficient sociable numbers has density zero. The set of even abundant sociable numbers has density zero. Finally, the set of odd abundant numbers has density $\approx 1/500$.

Thank you for your attention!