PRACTICAL PRETENDERS

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Abstract. Following Srinivasan, an integer $n \geq 1$ is called *practical* if every natural number in $[1, n]$ can be written as a sum of distinct divisors of n. This motivates us to define $f(n)$ as the largest integer with the property that all of $1, 2, 3, \dots, f(n)$ can be written as a sum of distinct divisors of n. (Thus, n is practical precisely when $f(n) \geq n$.) We think of $f(n)$ as measuring the "practicality" of n; large values of f correspond to numbers n which we term *practical pretenders*. Our first theorem describes the distribution of these impostors: Uniformly for $4 \leq y \leq x$,

$$\#\{n \leq x : f(n) \geq y\} \asymp \frac{x}{\log y}.$$

This generalizes Saias's result that the count of practical numbers in $[1, x]$ is $\asymp \frac{x}{\log x}$.

Next, we investigate the maximal order of f when restricted to non-practical inputs. Strengthening a theorem of Hausman and Shapiro, we show that every $n > 3$ for which

$$f(n) \geq \sqrt{e^{\gamma} n \log\log n}$$

is a practical number.

Finally, we study the range of f. Call a number m belonging to the range of f an *additive endpoint*. We show that for each fixed $A > 0$ and $\epsilon > 0$, the number of additive endpoints in $[1, x]$ is eventually smaller than $x/(\log x)^A$ but larger than $x^{1-\epsilon}$.

1. Introduction

In 1948, Srinivasan [15] initiated the study of *practical numbers*, natural numbers n with the property that each of $1, 2, 3, \dots, n-1$ admits an expression as a sum of distinct divisors of n. For example, every power of 2 is practical (since every natural number admits a binary expansion), but there are many unrelated examples, such as $n = 6$ or $n = 150$. Srinivasan posed two problems: Classify all practical numbers and say something interesting about their distribution.

The first of these tasks was carried to completion by Stewart [16] in 1954. The same classification was discovered independently, and almost concurrently, by Sierpiński [14]. Given a natural number n, write its canonical prime factorization in the form

$$n := p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}, \quad \text{where} \quad p_1 < p_2 < \cdots < p_r. \tag{1.1}$$

Put $n_0 = 1$, and for $1 \leq j \leq r$, put $n_j := \prod_{i=1}^{j} p_i^{e_i}$. Using σ for the usual sum-of-divisors function (so that $\sigma(m) := \sum_{d \mid m} d$), the number n is practical if and only if

$$p_{j+1} \leq \sigma(n_j) + 1 \quad \text{for all} \quad 0 \leq j < r. \tag{1.2}$$

Below, we refer to this as the *Stewart–Sierpiński classification* of practical numbers. This criterion implies, in particular, that all practical numbers $n > 1$ are even. Stewart and Sierpiński also showed that if all of the inequalities (1.2) hold, then not only are all integers in $[1, n-1]$ expressible as a sum of distinct divisors of n, but the same holds for all integers in the longer interval $[1, \sigma(n)]$. Note that $[1, \sigma(n)]$ is the largest interval one could hope to represent, since the sum of all distinct divisors of n is $\sigma(n)$.

2010 *Mathematics Subject Classification.* Primary: 11N25, Secondary: 11N37.

Key words and phrases. practical number, panarithmic number, sum-of-divisors function.

The distribution of practical numbers has proved more recalcitrant. Let $P(x)$ denote the count of practical numbers not exceeding x. Already in 1950, Erdős [2] claimed he could show that the practical numbers have asymptotic density zero, i.e., that $P(x) = o(x)$ as $x \to \infty$, but he gave no details. In 1984, Hausman and Shapiro [6] made the more precise assertion that $P(x) \leq x/(\log x)^{\beta+o(1)}$, with $\beta = \frac{1}{2}(1 - 1/\log 2)^2 \approx 0.0979\dots$. Their proof has an error (specifically, [6, Lemma 3.2] is incorrect); one should replace β with the smaller exponent $1 - \frac{1+\log\log 2}{\log 2} \approx 0.0860713$. Much sharper results on $P(x)$ were soon established by Tenenbaum [17, 19], who proved that $P(x) = \frac{x}{\log x}(\log\log x)^{O(1)}$. By a refinement of Tenenbaum's methods, Saias [13] established in 1997 what is still the sharpest known result: There are absolute constants c_1 and c_2 with

$$c_1 \frac{x}{\log x} \leq P(x) \leq c_2 \frac{x}{\log x} \quad \text{for all} \quad x \geq 2. \tag{1.3}$$

On the basis of the numerical data, Margenstern [9] has conjectured that $\frac{P(x)}{x/\log x}$ tends to a limit ≈ 1.341.

In this paper, we are concerned with what we term *near-practical numbers* or *practical pretenders*. Define $f(n)$ as the largest integer with the property that all of the numbers $1, 2, 3, \dots, f(n)$ can be written as a sum of distinct divisors of n. By definition, n is practical precisely when $f(n) \geq n - 1$. We define a *near-practical* number as one for which $f(n)$ is "large". This definition is purposely vague; its nebulous nature suggests that we investigate the behavior of the two-parameter function

$$N(x, y) := \#\{n \leq x : f(n) \geq y\}$$

for all x and y. Our first result gives the order of magnitude of the near-practical numbers for essentially all interesting choices of x and y.

Theorem 1.1. *There are absolute positive constants c_3 and c_4 so that for $4 \leq y \leq x$, we have*

$$c_3 \frac{x}{\log y} \leq N(x, y) \leq c_4 \frac{x}{\log y}.$$

Remark. To see why the technical restriction $y \geq 4$ is necessary, note that $N(x, x) = 0$ for all $3 < x < 4$.

Theorem 1.1 has the following easy corollary, proved in §3.

Corollary 1.2. *For each m, the set of natural numbers n with $f(n) = m$ possesses an asymptotic density, say ρ_m. The constant ρ_m is positive whenever there is at least one n with $f(n) = m$. Moreover,*

$$\sum_{m=1}^{\infty} \rho_m = 1.$$

We call a natural number m for which ρ_m is nonvanishing (equivalently, an m in the image of f) an *additive endpoint*. Thus, Corollary 1.2 shows that ρ_m is the probability mass function for additive endpoints. The first several additive endpoints are

$$1, 3, 7, 12, 15, 28, 31, 39, 42, 56, 60, 63, 73, 90, 91, 96, 100, 104, 108, 112, 120, \dots.$$

Just from this limited data, one might conjecture that ρ_m is usually zero, i.e., zero apart from of a set of m of vanishing asymptotic density. This guess is confirmed, in a much sharper form, in our next theorem.

Theorem 1.3. *For each fixed $A > 0$ and all $x \ge 3$, the number of integers in $[1, x]$ which occur as additive endpoints is $\ll_A x/(\log x)^A$. In the opposite direction, the number of additive endpoints up to x exceeds*

$$x/\exp(c_5(\log\log x)^3)$$

for all large x, for some absolute constant $c_5 > 0$.

Above, we noted Stewart's result that if $f(n) \ge n-1$, then $f(n) = \sigma(n)$. In this statement, a weak lower estimate on $f(n)$ implies that $f(n)$ is as large as possible. Hausman and Shapiro [6] proposed investigating the extent of this curious phenomenon. More specifically, they asked for the slowest-growing monotone function $g(n)$ for which $f(n) \ge g(n)$ implies (at least for n large) that n is practical. Set

$$H(n) := \sqrt{e^\gamma n \log\log n},$$

where γ is the Euler–Mascheroni constant. The next proposition appears as [6, Theorems 2.1, 2.2].

Proposition 1.4. *Let $\epsilon > 0$. Apart from finitely many exceptional n, all solutions to $f(n) \ge (1+\epsilon)H(n)$ are practical. On the other hand, there are infinitely many non-practical n with $f(n) \ge (1-\epsilon)H(n)$.*

Our final theorem removes the factor $1+\epsilon$ from the first half of Proposition 1.4.

Theorem 1.5. *If $n > 3$ and $f(n) \ge H(n)$, then n is practical.*

Notation. We use the Landau–Bachmann o and O symbols, as well as Vinogradov's $\ll$ notation, with their usual meanings; subscripts indicate dependence of implied constants. We write $\omega(n) := \sum_{p|n} 1$ for the number of distinct prime factors of n and $\Omega(n) := \sum_{p^k|n} 1$ for the number of prime factors of n counted with multiplicity; $\Omega(n; y) := \sum_{p^k|n,\ p\le y} 1$ denotes the number of prime divisors of n not exceeding y, again counted with multiplicity. The number of divisors of n is denoted $d(n)$. We use $P^-(m)$ for the smallest prime factor of m, with the convention that $P^-(1)$ is infinite. Absolute positive constants are denoted by c_1, c_2, c_3, etc., and have the same meaning each time they appear.

2. Proofs of Theorem 1.1 and Corollary 1.2

We begin by recording some useful lemmas. Our first gives a formula for $f(n)$ in terms of the prime factorization of n.

We assume that the factorization of n has been given in the form (1.1). We define $n_0 := 1$ and $n_j := \prod_{1\le i\le j} p_i^{e_i}$. Let $0 \le j < r$ be the first index for which $p_{j+1} > \sigma(n_j)+1$, putting $j = r$ if no such index exists (i.e., if n is practical). Then n_j is a practical number, by the Stewart–Sierpiński classification, and we call n_j the *practical component* of n.

Lemma 2.1. *We have $f(n) = \sigma(n_j)$, where n_j is the practical component of n.*

Proof. Since n_j is practical, $f(n) \ge f(n_j) = \sigma(n_j)$. On the other hand, $\sigma(n_j)+1$ is not representable as a sum of proper divisors of n. Indeed, if d is a divisor of n involved in an additive representation of $\sigma(n_j)+1$, then $d \le \sigma(n_j)+1 < p_{j+1} < p_{j+2} < \cdots < p_r$. It follows that the only primes dividing d are $p_1, \dots, p_j$, so that d is a divisor of n_j. But the largest number which can be formed as a sum of distinct divisors of n_j is $\sigma(n_j)$, which is smaller than $\sigma(n_j)+1$. So $\sigma(n_j)+1$ is not representable as a sum of distinct divisors of n, and hence $f(n) = \sigma(n_j)$, as claimed. □

The following lemma was observed by Margenstern [9, Corollaire 1] to follow from the Stewart–Sierpiński classification.

Lemma 2.2. *If n is practical and $m \le \sigma(n)+1$, then mn is practical.*

We now employ Lemma 2.2 to show that reasonably short intervals contain a positive proportion of practical numbers.

Lemma 2.3. *Let $\epsilon > 0$. For $x > x_0(\epsilon)$, the number of practical numbers in $((1-\epsilon)x, x]$ is $\gg_\epsilon x/\log x$.*

Proof. We can assume that $0 < \epsilon < 1$. With c_1 and c_2 as defined in (1.3), we set $r := \lceil 2c_2/c_1 \rceil$ and $s := \lceil 1/\epsilon \rceil$. From (1.3), we have that for large x (depending on ϵ), the number of practical numbers in the interval $(x/rs, x/s]$ is

$$\ge c_1 \frac{x/s}{\log(x/s)} - c_2 \frac{x/rs}{\log(x/rs)} > \frac{c_1}{3s}\frac{x}{\log x} \ge \frac{c_1}{6}\epsilon \frac{x}{\log x}.$$

By the pigeonhole principle, one of the intervals $(\frac{x}{s+1}, \frac{x}{s}]$, $(\frac{x}{s+2}, \frac{x}{s+1}]$, $\dots$, $(\frac{x}{rs}, \frac{x}{rs-1}]$ contains $> \frac{c_1}{6rs}\epsilon x/\log x \gg \epsilon^2 x/\log x$ practical numbers. Suppose this interval is $(\frac{x}{j+1}, \frac{x}{j}]$, where $s \le j < rs$, and let n be a practical number contained within. If $x > (rs)^2$, then $j < \frac{x}{j+1} < n$, and so jn is practical by Lemma 2.2. (Note that the lower bound on x assumed here depends only on ϵ.) Letting n run through the practical numbers in $(\frac{x}{j+1}, \frac{x}{j}]$, we obtain $\gg \epsilon^2 x/\log x$ practical numbers $jn \in (x\frac{j}{j+1}, x]$. But $(x\frac{j}{j+1}, x] \subset ((1-\epsilon)x, x]$, by our choice of s. This proves Lemma 2.3. Moreover, we have shown that the implied constant in the lemma statement may be taken proportional to ϵ^2. □

The next result, due to Hausman and Shapiro [6, Theorem 4.1], shows that substantially shorter intervals than those considered in Lemma 2.3 always contain at least one practical number.

Lemma 2.4. *For all real $x \ge 1$, there is a practical number $x < n < x + 2x^{1/2}$.*

Let $\Phi(x, y)$ denote the number of natural numbers $n \le x$ divisible by no primes $\le y$. The following lemma is a consequence of Brun's sieve. Variants can be found, e.g., as [3, Theorem 1, p. 201] or [18, Theorem 3, p. 400].

Lemma 2.5. *Uniformly for $2 \le y \le x$, we have $\Phi(x, y) \ll x/\log y$. If we assume also that $x > c_6 y$ for a suitable large absolute constant c_6, then $\Phi(x, y) \gg x/\log y$.*

We now prove Theorem 1.1, treating the upper and lower estimates separately.

Proof of the upper bound in Theorem 1.1. Suppose that $n \le x$ and $f(n) \ge y$. By the upper bound in (1.3), we may restrict our attention to non-practical n. Let d be the practical component of n and write $n = dq$. By Lemma 2.1, $\sigma(d) = f(n)$. In particular, since we are assuming that $f(n) \ge y \ge 4$, we must have that $d > 1$. Moreover, since n is not practical, $d < n$. Thus, $q > 1$ and

$$P^-(q) > \sigma(d) + 1 > d.$$

Hence,

$$d^2 < d \cdot P^-(q) \le dq = n \le x,$$

and so $d \le \sqrt{x}$.

Given d, the number of possibilities for n is bounded above by the number of $q \le x/d$ with $P^-(q) > d$. Since $2 \le d \le x/d$, we may apply Lemma 2.5 to find that the number of possibilities for q is $\ll \frac{x}{d \log d}$. Since $\sigma(d) = f(n) \ge y$ and (crudely) $\sigma(d) < d^2$, it follows that $d > \sqrt{y}$. Hence, using partial summation and (1.3), we see that the number of possibilities for n is

$$\ll x \sum_{\substack{\sqrt{y}<d\le\sqrt{x}\\ d \text{ practical}}} \frac{1}{d\log d} \le x\frac{P(\sqrt{x})}{\sqrt{x}\log\sqrt{x}} + x\int_{\sqrt{y}}^{\sqrt{x}} P(t)\frac{1+\log t}{(t\log t)^2}\,dt$$

$$\ll \frac{x}{(\log x)^2} + x\int_{\sqrt{y}}^{\sqrt{x}} \frac{dt}{t(\log t)^2} \ll \frac{x}{(\log x)^2} + \frac{x}{\log y} \ll \frac{x}{\log y}. \qquad \square$$

Proof of the lower bound in Theorem 1.1. The proof is suggested by that offered for the upper bound, but some care is necessary to ensure uniformity throughout the stated range of x and y.

First, we treat the range when $x^{1/10} \le y \le x$. In this domain, we use the trivial lower bound $N(x,y) \ge N(x,x)$. We estimate the right-hand side from below by counting practical numbers n belonging to the interval $[\frac{x+1}{2}, x]$. Note that for such n, we have $f(n) = \sigma(n) \ge 2n-1 \ge x$ (using for the first inequality that $n-1$ is a sum of proper divisors of n), and so n is indeed counted by $N(x,x)$.

If $6 \le x \le 11$, then $n = 6$ is a practical number in $[\frac{x+1}{2}, x]$. Similarly, if $4 \le x \le 6$, then $n = 4$ works. Finally, if $x \ge 11$, then Lemma 2.4 gives a practical number n with

$$\frac{x+1}{2} < n < \frac{x+1}{2} + 2\sqrt{\frac{x+1}{2}} \le x.$$

Hence, we always have $N(x,x) \ge 1$. (Recall that we only consider $x \ge 4$.) Moreover, by Lemma 2.3, there are $\gg x/\log x$ practical numbers in $[\frac{x+1}{2}, x]$ once x is large. It follows that $N(x,x) \gg x/\log x$ for all $x \ge 4$. So if $x^{1/10} \le y \le x$, then

$$N(x,y) \ge N(x,x) \gg x/\log x \gg x/\log y,$$

which gives the lower bound of the theorem in this case.

Now suppose that $4 \le y < x^{1/10}$. We consider numbers of the form $n = dq \le x$, where d is a practical number in $(y, y^3]$ and where $P^-(q) > y^6$. For any such n, we have $f(n) \ge f(d) \ge d > y$. Moreover, each n constructed in this way arises exactly once, since q is determined as the largest divisor of n supported on primes $> y^6$. Given d, the number of corresponding q is $\Phi(x/d, y^6)$. If x is large, then

$$\frac{x/d}{y^6} \ge \frac{x}{y^9} \ge x^{1/10} > c_6,$$

and so Lemma 2.5 gives

$$\Phi(x/d, y^6) \gg \frac{x}{d\log y}. \tag{2.1}$$

On the other hand, (2.1) is trivial for bounded x, since 1 is always counted by $\Phi(x/d, y^6)$. Thus, (2.1) holds in any case. Hence, the number of n constructed in this way is

$$\gg \frac{x}{\log y} \sum_{\substack{y<d\le y^3\\ d \text{ practical}}} \frac{1}{d}.$$

That the sum appearing here is $\gg 1$ for large y follows from partial summation and the lower bound in (1.3). For bounded y, the sum is also $\gg 1$, since Lemma 2.4 guarantees that there is at least one practical number between y and y^3. (Certainly $y + 2y^{1/2} < 3y < y^3$ when $y \ge 4$.) This completes the proof of the lower bound. $\square$

Proof of Corollary 1.2. One can detect whether or not $f(n) = m$ given just the list of divisors of n not exceeding $m+1$. Thus, whether or not $f(n) = m$ depends only on the residue class of n modulo $(m+1)!$. This gives the first two assertions of the corollary. For the third, notice that $1 - \sum_{n=1}^{N} \rho_m$ represents the density of the set of n with $f(n) > N$, which is $\ll 1/\log N$ by Theorem 1.1. Letting $N \to \infty$ completes the proof. □

3. Proof of Theorem 1.3

We divide the proof of Theorem 1.3 into two parts.

3.1. The upper bound in Theorem 1.3.

Central to the proof of both halves of Theorem 1.3 is the observation, immediate from Lemma 2.1, that m belongs to the range of f precisely when $m = \sigma(n)$ for some practical number n. Thus, we are really asking in Theorem 1.3 for estimates on the range of σ restricted to practical inputs.

Lemma 3.1. *Let $A \geq 30$. Suppose that $x \geq 3$. If n is a practical number with $x^{3/4} < n \leq x$, then either*

$$\Omega(n) > 2A \log\log x \tag{3.1}$$

or

$$\omega(n) > \frac{1}{2\log A} \log\log x. \tag{3.2}$$

Proof. Since n is practical, every integer in $[1, n]$ can be written as a subset-sum of divisors of n. Thus, $2^{d(n)} \geq n$, so we can use the hypothesis that $n > x^{3/4}$ to show

$$d(n) \geq \frac{\log n}{\log 2} > \frac{3/4}{\log 2} \log x > \log x.$$

Suppose that $n = \prod_{i=1}^{\ell} p_i^{e_i}$ is the factorization of n into primes, where $\ell = \omega(n)$. Since $d(n) = \prod_{i=1}^{\ell}(e_i + 1) > \log x$, the inequality between the arithmetic and geometric means gives that

$$\frac{1}{\ell^\ell} \left(\sum_{i=1}^{\ell} (e_i + 1) \right)^{\ell} \geq \prod_{i=1}^{\ell} (e_i + 1) > \log x. \tag{3.3}$$

Now assume that (3.1) fails. Then $\sum_{i=1}^{\ell}(e_i + 1) \leq 2\sum_{i=1}^{\ell} e_i \leq 4A \log\log x$, and (3.3) gives $(\frac{4A \log\log x}{\ell})^{\ell} > \log x$. Writing $\ell = \lambda \log\log x$, we deduce that

$$\left(\frac{4A}{\lambda}\right)^{\lambda \log\log x} > \log x, \quad \text{and so} \quad \lambda \log \frac{4A}{\lambda} > 1.$$

This latter inequality, along with the condition $A \geq 30$, implies that $\lambda > \frac{1}{2\log A}$ (by a short exercise in calculus). Since $\omega(n) = \lambda \log\log x$, we have (3.2). □

The next lemma, which belongs to the study of the anatomy of integers, bounds from above the number of n with an abnormally large number of small prime factors.

Lemma 3.2. *Let $x, y \geq 2$, and let $k \geq 1$. The number of $n \leq x$ with $\Omega(n; y) \geq k$ is $\ll \frac{k}{2^k} x \log y$.*

Remark. As a special case (when $y = x$), the number of $n \leq x$ with $\Omega(n) \geq k$ is $\ll \frac{k}{2^k} x \log x$.

Proof. The proof is almost identical to that suggested in Exercise 05 of [4, p. 12], details of which can be found in [8, Lemmas 12, 13]. Thus, we only sketch it. Let $v := 2 - 1/k$. Let g be the arithmetic function determined through the convolution identity $v^{\Omega(n;y)} = \sum_{d|n} g(d)$. Then g is multiplicative. For $e \geq 1$, we have $g(p^e) = v^e - v^{e-1}$ if $p \leq y$, and $g(p^e) = 0$ if $p > y$. Hence,

$$\sum_{n \leq x} v^{\Omega(n;y)} = \sum_{d \leq x} g(d) \left\lfloor \frac{x}{d} \right\rfloor \leq x \sum_{d \leq x} \frac{g(d)}{d}$$
$$\leq x \prod_{p \leq y} \left(1 + \frac{v-1}{p} + \frac{v^2 - v}{p^2} + \dots\right) = \frac{x}{2-v} \prod_{3 \leq p \leq y} \left(1 + \frac{v-1}{p-v}\right).$$

Now $2 - v = 1/k$, and the rightmost product is

$$\leq \exp\left(\sum_{3 \leq p \leq y} \frac{v-1}{p-v}\right) \leq \exp\left(\sum_{3 \leq p \leq y} \frac{1}{p-2}\right) \leq \exp\left(\sum_{p \leq y} \frac{1}{p} + O(1)\right) \ll \log y.$$

Collecting our estimates, we have shown that

$$\sum_{n \leq x} v^{\Omega(n;y)} \ll kx \log y.$$

But each term with $\Omega(n; y) \geq k$ makes a contribution to the left-hand side that is $\geq v^k = (2 - 1/k)^k = 2^k(1 - \frac{1}{2k})^k \gg 2^k$. Thus, the number of such terms is $\ll \frac{k}{2^k} x \log y$. □

The next lemma is a partial shifted-primes analogue of the Hardy-Ramanujan inequalities. A proof can be found in the text of Prachar [11, Lemma 7.1, p. 166] (cf. Erdős [1]). There a slightly stronger assertion is shown for shifted primes $p - 1$; only trivial changes are required to replace $p - 1$ with $p + 1$.

Lemma 3.3. *Let $t \geq 3$, and let $k \geq 1$. The number of primes $p \leq t$ with $\omega(p + 1) = k$ is*

$$\ll \frac{t}{(\log t)^2} \left(\frac{(\log \log t + c_7)^{k+2}}{(k-1)!} + 1\right).$$

Proof of the upper bound in Theorem 1.3. It is enough to prove the result for large values of A. Suppose that $m \leq x$ is an additive endpoint, and write $m = \sigma(n)$ with n practical. Put $Z := 2A \log \log x$. The number of values of m corresponding to an integer $n \leq x^{3/4}$ or an n with $\Omega(n) > Z$ is, by Lemma 3.2,

$$\ll x^{3/4} + \frac{Z}{2^Z} x \log x \ll_A \frac{x}{(\log x)^A}.$$

Thus, with

$$Z' := \frac{1}{2 \log A} \log \log x,$$

Lemma 3.1 allows us to assume that

$$\omega(n) \geq Z'. \tag{3.4}$$

We now show that most of the primes dividing n make a large contribution to $\Omega(\sigma(n)) = \Omega(m)$. We claim we can assume that both of the following hold:

(i) There are fewer than $Z'/4$ primes p for which $p^2 \mid n$.
(ii) There are fewer than $Z'/4$ primes p dividing n for which

$$\Omega(p + 1) \leq 8A \log A. \tag{3.5}$$

With $K := \lceil Z'/4 \rceil$, the number of $n \le x$ which are exceptions to (i) is, by the multinomial theorem,

$$\le x \sum_{\substack{d \le x,\ \text{squarefree} \\ \omega(d) = K}} \frac{1}{d^2} \le \frac{x}{K!} \left(\sum_{p \le x} \frac{1}{p^2} \right)^K \le x(e/K)^K < x/(\log x)^A,$$

once x is large. (We use here that $\sum p^{-2} < 1$ and the elementary inequality $K! \ge (K/e)^k$.) To handle (ii), we observe that from Lemma 3.3 and partial summation, the sum of the reciprocals of all p satisfying (3.5) converges. Let S denote this sum. Then the number of exceptions to (ii) is, for large x,

$$\le \frac{x}{K!} \Bigg(\sum_{\substack{p \le x \\ p \text{ satisfies (3.5)}}} \frac{1}{p} \Bigg)^K \le x(eS/K)^K < x/(\log x)^A.$$

Hence, we can indeed assume (i) and (ii).

From (3.4), it now follows that there are at least $Z' - 2\frac{Z'}{4} = \frac{Z'}{2}$ primes p for which $p \parallel n$ and for which $\Omega(p+1) > 8A \log A$. Hence,

$$\Omega(m) = \Omega(\sigma(n)) \ge \sum_{p \parallel n} \Omega(p+1) > 8A \log A \cdot \frac{Z'}{2} = 2A \log\log x.$$

But by Lemma 3.2, the number of $m \le x$ with $\Omega(m)$ this large is $\ll_A x/(\log x)^A$. This completes the proof of Theorem 1.3 for large x. If x is bounded in terms of A, then the theorem is trivial. □

Remark. The method given here can be pushed to yield the more explicit result that the count of $m \le x$ that occur as additive endpoints is smaller than

$$x/\exp\left(c_8 \log\log x \frac{\log\log\log x}{\log\log\log\log x} \right).$$

3.2. **The lower bound in Theorem 1.3.** The lower bound in Theorem 1.3 will be deduced from the following proposition, which may be of interest outside of this context.

Proposition 3.4. *Let $A > 0$. There is a constant $c = c(A)$ so that the following holds. If x is sufficiently large, say $x > x_0(A, c)$, then any subset $\mathscr{S} \subset [1, x]$ with*

$$\#\mathscr{S} \le x/\exp(c(\log\log x)^3)$$

satisfies

$$\#\sigma^{-1}(\mathscr{S}) \le x/(\log x)^A.$$

Here $\sigma^{-1}(\mathscr{S})$ denotes the set of n with $\sigma(n) \in \mathscr{S}$.

Remark. It is perhaps surprising that one cannot improve the upper bound on $\#\sigma^{-1}(\mathscr{S})$ very much, even if one assumes that $\mathscr{S}$ consists of only a single element! Indeed, plausible conjectures about the distribution of smooth shifted primes $p+1$ (such as what would follow from the Elliott–Halberstam conjecture) imply that for all large x, there is a singleton set $\mathscr{S} \subset [1, x]$ with $\#\sigma^{-1}(\mathscr{S}) > x^{1-\epsilon}$. (Here $\epsilon > 0$ is arbitrary but fixed.) For the Euler φ-function, this result is due to Erdős [1] (see also the exposition of Pomerance [10]); the σ-version can be proved similarly, replacing $p-1$ with $p+1$ when necessary.

To apply Proposition 3.4 to the case of the practical numbers, it is convenient to recall Gronwall's determination of the maximal order of the sum-of-divisors function σ [5, Theorem 323, p. 350].

Lemma 3.5. *We have* $\limsup_{n\to\infty} \frac{\sigma(n)}{n\log\log n} = e^{\gamma}$.

Proof of the lower bound in Theorem 1.3. Let x be large. By Lemma 3.5, if $n \le \frac{x}{2\log\log x}$, then $\sigma(n) \le x$. (We use here that $e^{\gamma} < 2$.) Thus, with $\mathscr{S}$ the set of additive endpoints not exceeding x,

$$\#\sigma^{-1}(\mathscr{S}) \ge PR\left(\frac{x}{2\log\log x}\right) \gg \frac{x}{(\log x)(\log\log x)},$$

using the lower estimate in (1.3) for the last step. The desired lower bound on $\#\mathscr{S}$ now follows from (the contrapositive of) Proposition 3.4, with $A = 1.1$. □

The rest of this section is devoted to the proof of Proposition 3.4. The proof rests on a σ-analogue of a result for the Euler function appearing in a paper of Luca and the first author [7, Lemma 2.1].

Lemma 3.6. *Let $x \ge 3$. Let d be a squarefree natural number with $d \le x$. The number of n for which $d \mid \sigma(n)$ and $\sigma(n) \le x$ is*

$$\le \frac{x}{d}(c_9 \log x)^{3\omega(d)}.$$

Proof. If $d = 1$, the result is clear. Suppose that $d > 1$. Let n be an integer for which $\sigma(n) \in [1, x]$ is a multiple of d. Write the prime factorization of n in the form $n = \prod_i p_i^{e_i}$. Since $d \mid \sigma(n)$, there is a factorization $d = d_1 d_2 \cdots$ for which each $d_i \mid \sigma(p_i^{e_i})$. Discarding those terms with $d_i = 1$ and relabeling, we can assume that $d = d_1 \cdots d_\ell$, where each $d_i > 1$. Clearly, $\ell \le \omega(d)$.

We now fix the factorization $d = d_1 \cdots d_\ell$ and count the number of corresponding n. This count does not exceed

$$x \prod_{i=1}^{\ell} \left(\sum_{\substack{p^e:\ \sigma(p^e)\le x \\ d_i \mid \sigma(p^e)}} \frac{1}{p^e} \right). \tag{3.6}$$

We proceed to estimate the inner sum in (3.6). If $d_i \mid \sigma(p^e)$, then $\sigma(p^e) = d_i m$, with $m \le x/d_i$. Since $\sigma(p^e) = 1 + p + \cdots + p^e \le 2p^e$,

$$\sum_{\substack{p^e:\ \sigma(p^e)\le x \\ d_i \mid \sigma(p^e)}} \frac{1}{p^e} \le \frac{2}{d_i} \sum_{m \le x/d_i} \frac{1}{m} \sum_{p^e:\ \sigma(p^e) = md_i} 1.$$

For each fixed $e \ge 1$, there is at most one prime p with $\sigma(p^e) = md_i$; moreover, since $md_i \le x$, there are no such p once $e > \log x/\log 2$. Thus,

$$\frac{2}{d_i} \sum_{m \le x/d_i} \frac{1}{m} \sum_{p^e:\ \sigma(p^e) = md_i} 1 \ll \frac{\log x}{d_i} \sum_{m \le x/d_i} \frac{1}{m} \ll \frac{(\log x)^2}{d_i}.$$

Inserted back into (3.6), we find that for a certain absolute constant $C > 1$, the number of n corresponding to the given factorization is at most

$$x \prod_{i=1}^{\ell} \frac{C(\log x)^2}{d_i} = \frac{x}{d} C^{\ell} (\log x)^{2\ell} \le \frac{x}{d} C^{\omega(d)} (\log x)^{2\omega(d)}.$$

Finally, we sum over unordered factorizations of d into parts > 1. Since d is squarefree, the number of such factorizations is precisely $B_{\omega(d)}$, where B_k denotes the kth Bell number (the number of set partitions of a k-element set). Thinking combinatorially, we have the crude bound $B_k \le k^k$, and so the total number of n which arise is

$$\le \omega(d)^{\omega(d)} \left(\frac{x}{d} C^{\omega(d)} (\log x)^{2\omega(d)}\right) = \frac{x}{d} (C\omega(d)(\log x)^2)^{\omega(d)}.$$

By definition, we have $\omega(d) \le \Omega(d) \le \log x / \log 2$, where the final inequality follows from the simple observation that $2^{\Omega(d)} \le d \le x$. This proves our lemma with $c_9 = (C/\log 2)^{1/3}$. □

Lemma 3.7. *Fix $A \ge 3$. The number of $n \le x$ for which*

$$\Omega(\sigma(n)) \ge 8A^2 (\log\log x)^2 \tag{3.7}$$

is $o(x/(\log x)^A)$, as $x \to \infty$.

Proof. We may suppose that $\omega(n) \le 2A \log\log x$. Indeed, Lemma 3.2 shows that for $x \ge 3$, the number of $n \le x$ not satisfying the stronger inequality $\Omega(n) \le 2A \log\log x$ is

$$\ll \frac{A \log\log x}{2^{A \log\log x}} x \log x \ll_A \frac{x \log\log x}{(\log x)^{2A \log 2 - 1}}.$$

Since $A \ge 3$, the exponent $2A \log 2 - 1 > A$, and so this upper bound is $o(x/(\log x)^A)$.

Writing $\Omega(\sigma(n)) = \sum_{p^e \| n} \Omega(\sigma(p^e))$, we thus deduce that if (3.7) holds, then

$$\Omega(\sigma(p^e)) \ge \frac{8A^2(\log\log x)^2}{2A \log\log x} = 4A \log\log x \tag{3.8}$$

for some prime power $p^e \parallel n$.

Suppose first that $e > 1$. Then (for large x) the squarefull part of n is of size at least

$$p^e \ge \frac{1}{2}\sigma(p^e) \ge \frac{1}{2} 2^{\Omega(\sigma(p^e))} \ge \frac{1}{2} 2^{4A \log\log x} > (\log x)^{5A/2}. \tag{3.9}$$

But then the number of possibilities for $n \le x$ is $\ll x/(\log x)^{5A/4}$, and so in particular is $o(x/(\log x)^A)$. On the other hand, if $e = 1$, then (3.8) implies that n is divisible by some prime p with $\Omega(p+1) \ge 4A \log\log x$. For each such p, the number of corresponding n is $\le x/p < 2x/(p+1)$. Summing over p, we find that the total number of such $n \le x$ is at most

$$2x \sum_{\substack{d \le x \\ \Omega(d) \ge 4A \log\log x}} \frac{1}{d}.$$

Put $Z := 4A \log\log x$; by partial summation, along with Lemma 3.2 and the final inequality in (3.9), this upper bound is

$$\ll x \frac{Z}{2^Z} \int_2^x \frac{\log t}{t}\, dt \ll x (\log x)^2 \frac{Z}{2^Z} \ll_A x \frac{(\log x)^2 \log\log x}{(\log x)^{5A/2}},$$

and so is $o(x/(\log x)^A)$, as $x \to \infty$. This completes the proof. □

Lemma 3.8. *Let $x \ge 3$, and let $z \ge 1$. The number of $n \le x$ with $\sigma(n)$ divisible by p^2 for some prime $p > z$ is $\ll x(\log x)^2 z^{-1/2}$.*

Proof. If $p^2 \mid \sigma(n)$, then either $p \mid \sigma(q^e)$ for a proper prime power q^e exactly dividing n, or there are two distinct primes q_1 and q_2 exactly dividing n with $q_1, q_2 \equiv -1 \pmod{p}$. In the former case, n has a squarefull divisor of size $\ge q^e \ge \frac{1}{2}\sigma(q^e) > p/2 > z/2$. The number of such n is $\ll xz^{-1/2}$, which is acceptable for us. For a given p, the number of n arising in the second case is

$$\leq x\Bigg(\sum_{\substack{q\leq x\\ q\equiv -1 \pmod p}} \frac{1}{q}\Bigg)^2 \leq x\left(\sum_{j\leq x}\frac{1}{pj-1}\right)^2 \ll x(\log x)^2 p^{-2}.$$

Summing over $p>z$, we find that the total number of n that can arise from this case is $\ll x(\log x)^2 z^{-1}$, which is also acceptable. □

Proof or Proposition 3.4. We may suppose that our fixed constant A satisfies $A\geq 5$. We will show that for such A, the proposition holds with $c(A)=50A^3$.

Let $\mathscr{S}_1$ consist of those $m\in\mathscr{S}$ for which either

(i) $m\leq x/(\log x)^{2A}$, or
(ii) $\Omega(m)\geq 8A^2(\log\log x)^2$, or
(iii) $p^2\mid m$ for some $p>(\log x)^{3A}$.

We let $\mathscr{S}_2$ consist of the remaining elements of $\mathscr{S}$. By Lemmas 3.7 and 3.8, the size of $\sigma^{-1}(\mathscr{S}_1)$ is $o(x/(\log x)^A)$ as $x\to\infty$ (uniformly in the choice of $\mathscr{S}$).

We turn now to $\mathscr{S}_2$. To each $m\in\mathscr{S}_2$, we associate the divisor m' of m defined by

$$m' := \prod_{\substack{p^e\| m\\ p>(\log x)^{3A}}} p^e.$$

Then m' is squarefree, and

$$\omega(m')\leq\Omega(m)<8A^2(\log\log x)^2. \tag{3.10}$$

Moreover, assuming that x is large, since $m>x/(\log x)^{2A}$,

$$m'\geq m/((\log x)^{3A})^{\Omega(m)} > \frac{x/(\log x)^{2A}}{\exp(24A^3(\log\log x)^3)} > x/\exp(25A^3(\log\log x)^3). \tag{3.11}$$

We bound the number of σ-preimages of m from above by the number of n for which $\sigma(n)\in[1,x]$ is a multiple of m'. By Lemma 3.6, along with (3.10) and (3.11), the number of such n is

$$\begin{aligned}\leq \frac{x}{m'}(c_9\log x)^{3\omega(m')} &\leq \exp(25A^3(\log\log x)^3)(c_9\log x)^{24A^2(\log\log x)^2}\\ &\leq \exp(49A^3(\log\log x)^3),\end{aligned}$$

say. Summing over the elements of $\mathscr{S}_2$, we find that

$$\#\sigma^{-1}(\mathscr{S}_2)\leq\exp(49A^3(\log\log x)^3)\cdot\#\mathscr{S}_2.$$

So if we assume that $\#\mathscr{S}\leq x/\exp(50A^3(\log\log x)^3)$, then $\#\sigma^{-1}(\mathscr{S}_2)=o(x/(\log x)^A)$, as $x\to\infty$. Combined with our earlier estimate on the size of $\sigma^{-1}(\mathscr{S}_1)$, this shows that $\#\sigma^{-1}(\mathscr{S})\leq x/(\log x)^A$ once x is sufficiently large. □

4. Proof of Theorem 1.5

The key to the proof of Theorem 1.5 is the following inequality of Robin [12, Théorème 2].

Lemma 4.1. *For each natural number $n\geq 3$,*

$$\sigma(n)\leq e^{\gamma} n\log\log n + 0.6483\frac{n}{\log\log n}.$$

Proof of Theorem 1.5. Suppose for the sake of contradiction that $f(n) \geq H(n)$ but that n is not practical. We assume to begin with that $n > 14$, treating small n at the end of the proof. Let d be the practical component of n, and write $n = dq$. Then $q > 1$, and

$$P^-(q) > \sigma(d) + 1 = f(n) + 1 > H(n) > n^{1/2},$$

where in fact the last inequality holds for all $n > 6$. It follows that q is prime and $P^-(q) = q$. Hence, $H(n) < q = n/d$, and so

$$d < \frac{n}{H(n)}.$$

Also, since $\sigma(d) = f(n) \geq H(n)$, we have

$$q = n/d \leq \frac{n}{d}\frac{\sigma(d)}{H(n)}.$$

Multiplying the last two displayed inequalities shows that

$$n = dq \leq \frac{\sigma(d)}{d}\left(\frac{n}{H(n)}\right)^2 = \frac{\sigma(d)}{d} n(e^\gamma \log\log n)^{-1},$$

and so

$$\frac{\sigma(d)}{d} \geq e^\gamma \log\log n. \tag{4.1}$$

Since $q > n^{1/2}$ and $n = dq$, we have that $q > d$, and so

$$\log\log n = \log\log(qd) > \log\log(d^2) = \log\log d + \log 2;$$

thus, (4.1) gives

$$\begin{aligned}\frac{\sigma(d)}{d} &\geq e^\gamma \log\log d + e^\gamma \log 2\\ &> e^\gamma \log\log d + 1.2345.\end{aligned} \tag{4.2}$$

We now derive a contradiction to Robin's inequality. We can assume that $d \geq 6$; otherwise, $\sigma(d)/d \leq 7/4$, and (4.1) then implies that $n \leq 14$, contrary to hypothesis. By Lemma 4.1,

$$\frac{\sigma(d)}{d} \leq e^\gamma \log\log d + \frac{0.6483}{\log\log d}.$$

Combining this inequality with (4.2), we obtain $0.6483/\log\log d > 1.2345$. But this fails for all $d \geq 6$. This contradiction completes the proof for $n > 14$.

It remains to treat the cases when $3 < n \leq 14$. For odd $n > 3$, the hypotheses of the theorem are never satisfied, since $f(n) = 1 < H(5) \leq H(n)$. So the only possible exceptions to the theorem have n even. The non-practical even values of $n \leq 14$ are $n = 10$ and $n = 14$, and in both cases, $f(n) = 3 < H(n)$, so the theorem holds. □

Acknowledgements

We thank Greg Martin and Carl Pomerance for helpful conversations.

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