A REMARK ON SOCIABLE NUMBERS OF ODD ORDER

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Abstract. Write $s(n)$ for the sum of the proper divisors of the natural number n. We call n *sociable* if the sequence n, $s(n)$, $s(s(n))$, … is purely periodic; the period is then called the *order of sociability* of n. The ancients initiated the study of order 1 sociables (*perfect numbers*) and order 2 sociables (*amicable numbers*), and investigations into higher-order sociable numbers began at the end of the 19th century.

We show that if k is odd and fixed, then the number of sociable $n \leq x$ of order k is bounded by $x/(\log x)^{1+o(1)}$ as $x \to \infty$. This improves on the previously best-known bound of $x/(\log\log x)^{1/2+o(1)}$, due to Kobayashi, Pollack, and Pomerance.

1. Introduction

Write $s(n)$ for the sum of the proper divisors of n, so that $s(n) = \sigma(n) - n$. We write $s_0(n)$ for n, and if $s_{k-1}(n)$ is defined and positive, we put $s_k(n) := s(s_{k-1}(n))$. The natural number n is called *sociable* if for some $k \geq 1$, the numbers $n, s(n), \dots, s_{k-1}(n)$ are all distinct while $n = s_k(n)$. In this case the set $\{n, s(n), \dots, s_{k-1}(n)\}$ is called a *sociable cycle* and k is called the *order of sociability* of n. Observe that the sociable numbers of order 1 are precisely the perfect numbers, while those of order 2 are the amicable numbers. In [KPP09], it is shown (see [KPP09, Theorem 1]) that the count of sociable numbers in $[1, x]$ of order k is at most

$$x/\exp((1 + o(1))\sqrt{\log_3 x \log_4 x}),$$

if $k = o(\sqrt{\log_3 x \log_4 x}/\log_5 x)$. (Here $\log_1 x := \max\{1, \log x\}$ and for $j > 1$, $\log_j x := \max\{1, \log(\log_{j-1} x)\}$.) For sociable numbers of odd order, one can do a bit better. From [KPP09, Theorem 2], the number of sociable numbers in $[1, x]$ of odd order k is bounded by

$$x/(\log_2 x)^{1/2+o(1)},$$

if $k = o(\log_3 x/\log_5 x)$. Our purpose here is to further sharpen the upper bound when k is small and odd.

Theorem 1. *Let $x \geq 3$, and let k be an odd natural number. The number of sociable numbers of order k contained in $[1, x]$ is at most $x/(\log x)^{1+o(1)}$, as $x \to \infty$, uniformly for $k = o(\log_4 x)$.*

Computational results on sociable numbers are recorded in [Coh70], [Fla91], [MM91], [MM93], and [Moe]. There are currently 175 known sociable cycles of order > 2. Of these, only two have odd order, one having order 5 and the other order 9.

2000 *Mathematics Subject Classification.* Primary: 11A25, Secondary: 11N25.

This material is based upon work supported by the National Science Foundation under agreement No. DMS-0635607. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

Notation. For natural numbers d and n, we write $d \parallel n$ to mean that d is a unitary divisor of n, i.e., that $d \mid n$ and $\gcd(d, n/d) = 1$. If p is a prime, we write $v_p(n)$ for the p-adic order of n, defined so that $p^{v_p(n)} \parallel n$.

2. Proof of Theorem 1

The proof requires a few preliminaries. The first of these is due to Erdős (see [Erd46, Theorem 2], [KPP09, Theorem B]).

Lemma 1. *For $x > 0$, the number of $n \le x$ with $\sigma(n)/n > u$ is bounded by*

$$x/\exp(\exp((e^{-\gamma} + o(1))u))$$

as $u \to \infty$, uniformly in x. Here γ is the Euler–Mascheroni constant.

The next two results are taken from a recent preprint of Luca and Pomerance [LP].

Lemma 2 (cf. [LP, Corollary 1])**.** *For any $\lambda \in (0, 2]$ and $x \ge 3$, we have the estimate*

$$\#\{n \le x : v_2(\sigma(n)) \le \lambda \log\log x\} \ll \frac{x}{(\log x)^{1+\lambda\log 2 - \lambda \log\left(1+\frac{1+\sqrt{4\lambda+1}}{2\lambda}\right) - \frac{2\lambda}{1+\sqrt{4\lambda+1}}}}, \tag{1}$$

where the implied constant is absolute.

Lemma 3 (cf. [LP, Lemma 2])**.** *Let $x \ge 2$, $z \ge 2$, and let $\mathcal{P}$ be a set of odd primes contained in the interval $[1, z]$. The number of $n \le x$ for which $\sigma(n)$ is coprime to every element of $\mathcal{P}$ is bounded by*

$$\frac{x}{(\log x)^{1-g_{\mathcal{P}}}} \exp(O((\log z)^2)),$$

where

$$g_{\mathcal{P}} := \prod_{p \in \mathcal{P}} \frac{p-2}{p-1}$$

and the O-constant is absolute.

Actually both results are stated in [LP] with the Euler function φ in place of σ, but the proofs are trivially adapted to the σ-case. We will not need the full strength of Lemma 2 and require only the following easy consequence, corresponding to letting $\lambda \to 0$:

Lemma 4. *Let $x \ge 2$ and let r be a natural number. The number of $n \le x$ with $v_2(\sigma(n)) < r$ is bounded by $x/(\log x)^{1+o(1)}$, provided that $r = o(\log_2 x)$.*

The next lemma describes the property of sociable cycles of odd order which plays the key role in our argument. If $\mathcal{S}$ is a set of natural numbers, we write $\gcd(\mathcal{S})$ for the greatest common divisor of the elements of $\mathcal{S}$. We also write $\sigma(\mathcal{S})$ for the set $\{\sigma(m) : m \in \mathcal{S}\}$.

Lemma 5. *Let $\mathcal{C}$ be a sociable cycle of odd order greater than 1. Then $\gcd(\sigma(\mathcal{C}))$ divides $\gcd(\mathcal{C})$, except possibly if $2 \parallel \gcd(\sigma(\mathcal{C}))$, in which case $\frac{1}{2}\gcd(\sigma(\mathcal{C})) \mid \gcd(\mathcal{C})$.*

Proof. For notational simplicity, put $d = \gcd(\sigma(\mathcal{C}))$. For each element $m \in \mathcal{C}$, observe that $s(m) = \sigma(m) - m \equiv -m \pmod{d}$. Applying this observation with m successively replaced by $s(m), s_2(m), \dots$, we find that $s_j(m) \equiv (-1)^j m$, for every natural number $j \ge 1$. Now if we take j as the order of $\mathcal{C}$, this shows that $m \equiv -m \pmod{d}$, so that $d \mid 2m$. Since this holds for every $m \in \mathcal{C}$, we get that $d \mid 2\gcd(\mathcal{C})$. In particular, if d is odd, then d divides $\gcd(\mathcal{C})$, and whenever d is even, $d/2$ divides $\gcd(\mathcal{C})$.

It remains to show that if d is even and $4 \mid d$, then $d \mid \gcd(\mathcal{C})$. Suppose that $2^e \parallel d$, where $e \geq 2$. From the preceding paragraph, we have that $2^{e-1} \mid \gcd(\mathcal{C})$, and we would like to prove that $2^e \mid \gcd(\mathcal{C})$. Otherwise, there is some $m \in \mathcal{C}$ for which $2^{e-1} \parallel m$. In this case, since $2^e \mid d$, we have that $2^{e-1} \parallel \sigma(m) - m = s(m)$. Iterating, we find that 2^{e-1} is a unitary divisor of every element of $\mathcal{C}$. Consequently, $\sigma(2^{e-1}) \mid \sigma(\mathcal{C}) = d$. Since $\sigma(2^{e-1})$ is odd, we infer from the last paragraph that $\sigma(2^{e-1}) \mid \gcd(\mathcal{C})$. Thus $2^{e-1}\sigma(2^{e-1})$ divides every element of our cycle $\mathcal{C}$. But this impossible: Indeed, the number $2^{e-1}\sigma(2^{e-1})$ is always either perfect or abundant, since

$$\sigma(2^{e-1}\sigma(2^{e-1})) = \sigma(2^{e-1})\sigma(\sigma(2^{e-1})) \geq \sigma(2^{e-1})(1+\sigma(2^{e-1})) = 2(2^{e-1}\sigma(2^{e-1})).$$

It follows that every element of $\mathcal{C}$ is either perfect or abundant, which is clearly impossible when $\#\mathcal{C} > 1$. ☐

Proof of Theorem 1. We can assume that $k > 1$, since much stronger results are known about the distribution of sociable numbers of order 1 (perfect numbers); see [Wir59] for the best result in this direction.

Let $n \leq x$ be a sociable number of odd order k, and let $\mathcal{C}$ be the corresponding cycle. We can assume that $\mathcal{C} \subset [1, X]$, where $X = x(2\log_3 x)^k$. Otherwise, for some $0 \leq j < k$, we have $s_j(n) \leq x(2\log_3 x)^j$ but $s_{j+1}(n)/s_j(n) > 2\log_3 x$. In this case, the number of possibilities for $s_j(n)$ is $\ll x(2\log_3 x)^j/\log x$ by Lemma 1. Since (for a given value of k) the number $n = s_{k-j}(s_j(n))$ is determined by j and $s_j(n)$, the number of possibilities for n is $\ll kx(2\log_3 x)^j/\log x$. But both k and $(2\log_3 x)^k$ have the shape $(\log x)^{o(1)}$, and so this case presents us with at most $x/(\log x)^{1+o(1)}$ possible values of n.

The results of the last paragraph reduce the theorem to showing that the number of sociable cycles of length k contained in $[1, X]$ is bounded by $X/(\log X)^{1+o(1)}$. Put

$$r = \lfloor \sqrt{k \log_3 x} \rfloor, \quad \text{so that for large } x, \quad \log_3 x \geq r \geq \sqrt{\log_3 x} \geq 2. \tag{2}$$

If $v_2(\gcd(\sigma(\mathcal{C}))) < r$, then $\mathcal{C}$ contains a term m with $v_2(\sigma(m)) < r$. By Lemma 4, the number of possibilities for m (and so also for its cycle) is bounded by $X/(\log X)^{1+o(1)}$.

So we can assume that $2^r \mid \gcd(\sigma(\mathcal{C}))$. By Lemma 5, we have that

$$2^r \mid \gcd(\sigma(\mathcal{C})) \mid \gcd(\mathcal{C}). \tag{3}$$

Now we exploit the fact since $\#\mathcal{C} > 1$, it must be that $\gcd(\mathcal{C})$ is deficient (cf. the conclusion of the proof of Lemma 5). Suppose that p is an odd prime divisor of $\gcd(\mathcal{C})$. Since $2^r p \mid \gcd(\mathcal{C})$, it must be that $2^r p$ is deficient, which implies (after a short computation) that $p > 2^{r+1}$. So any odd prime divisor of $\gcd(\mathcal{C})$ exceeds 2^{r+1}, and now from (3), we deduce that the same is true for each odd prime divisor of $\gcd(\sigma(\mathcal{C}))$. Put

$$\mathcal{P} := \{p \text{ prime} : 2 < p \leq 2^{r+1}\}, \quad \text{and for each } m \in \mathcal{C}, \text{ define} \quad \mathcal{P}_m := \{p \in \mathcal{P} : p \nmid \sigma(m)\}.$$

Then $\mathcal{P} \subset \bigcup_m \mathcal{P}_m$, and so (in the notation of Lemma 3)

$$\prod_{m \in \mathcal{C}} g_{\mathcal{P}_m} \leq g_{\mathcal{P}} = \prod_{2 < p \leq 2^{r+1}} \frac{p-2}{p-1} \ll \frac{1}{\log(2^{r+1})} \ll \frac{1}{\sqrt{\log_3 x}},$$

using Mertens's theorem to estimate the last product. Consequently, there is an $m \in \mathcal{C}$ with

$$g_{\mathcal{P}_m} \ll (\log_3 x)^{-\frac{1}{2k}}.$$

The upper bound here is $o(1)$, since $k = o(\log_4 x)$. So from Lemma 3 (with $x = X$ and $z = 2^{r+1}$), the number of possibilities for m (and so for its cycle) is bounded by $X/(\log X)^{1+o(1)}$.

(Here we use the upper bound on r in (2).) Noting that the number of possibilities for the set $\mathcal{P}_m$ is bounded by

$$2^{\#\mathcal{P}} \leq 2^{2^{r+1}} \leq 2^{2^{\log_3 x+1}} = (\log X)^{o(1)},$$

the theorem follows. □

3. Concluding remarks

We close with the following problem, which in view of Theorem 1 may be tractable:

Problem: Prove that for each odd k, the sum of the reciprocals of the sociable numbers of order k converges.

This problem is open for every odd $k > 1$.

Acknowledgements

This research was conducted while the author enjoyed the hospitality of the Institute for Advanced Study.

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